Journal of Statistical Physics (2024) 191:81
https://doi.org/10.1007/s10955-024-03292-x

The High-Order Corrections of Discrete Harmonic Measures and Their Correction Constants

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Received: 8 November 2022 / Accepted: 12 June 2024

Abstract
By the dimension reduction idea, overshoot for random walks, coupling and martingale arguments, we obtain a simpler and easily computable expression for the first-order correction constant between discrete harmonic measures for random walks with rotationally invariant step distribution in $\mathbb{R}^d$ ($d \geq 2$) and the corresponding continuous counterparts. This confirms and extends a conjecture in Jiang and Kennedy (J Theor Probab 30(4):1424–1444, 2017), and simplifies the related expression of Wang et al. (Bernoulli 25(3):2279–2300, 2019). Furthermore, we propose a universality conjecture on high-order corrections for error estimation between generalized discrete harmonic measures and their continuous counterparts, which generalizes the universality conjecture of the first-order correction in Kennedy (J Stat Phys 164(1):174–189, 2016); and we prove this conjecture heuristically for the rotationally invariant case, and also provide several examples of second-order error corrections to check the conjecture by a numerical simulation argument.

Keywords Harmonic measure · Random walk · High-order correction · Overshoot · Coupling

Communicated by Hal Tasaki.

The project is supported partially by the NSFC (No. 12171410, No. 12001463), the Education Department of Hunan Province (No. 23B0178), Hu Xiang Gao Ceng Ci Ren Cai Ju Jiao Gong Cheng-Chuang Xin Ren Cai (No. 2019RS1057) and the Graduate Innovation Project of Xiangtan University (No. XDCX2023Y105). S. Yang is supported by FAPSP 2023/12652-4.

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Published online: 04 July 2024

Mathematics Subject Classification 60J10 · 82B41

1 Introduction

Universality is an important topic in statistical physics and probability theory. For instance, the central limit theorem (CLT) and Donsker's invariance principle are kinds of universality in probability theory. Based on Donsker's invariance principle, can further research be done on universality? Indeed, the first-order corrections for error estimation between discrete harmonic measures and their continuous counterparts happens to be precisely such a kind of universality problem.

Motivated by Kennedy [23] and Jiang and Kennedy [19] in $\mathbb{R}^2$ (and also Wang et al. [31] in $\mathbb{R}^d$ with $d \geq 2$), in this paper, we investigate the universality for the first or high-order corrections between discrete and continuous harmonic measures in $\mathbb{R}^d$ with $d \geq 1$.

To begin, denote $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^d$, $x = (x_1, \cdots, x_d)$ for any $x \in \mathbb{R}^d$, and $\mathbb{N} = \{1, 2, 3, \cdots\}$. Let $\{X_i\}_{i=1}^{\infty}$ be an i.i.d sequence of random variables in $\mathbb{R}^d$ with common rotationally invariant step distribution μ on unit open (or closed) ball $\mathbb{B}^d \in \mathbb{R}^d$ satisfying $\mu\{\mathbf{0}\} = 0$ (here $\mu\{\mathbf{0}\} = 0$ can be replaced by $\mu\{\mathbf{0}\} < 1$. Indeed, for any measurable subset A of $\mathbb{B}^d$, note that the random walks $\{\delta S_n^{\mu}\}_{n\geq 0}$ and $\left\{\delta S_n^{\widehat{\mu}}\right\}_{n\geq 0}$ have the same discrete harmonic measure if we replace $\widehat{\mu}$ by $\frac{1}{1-\mu(\{\mathbf{0}\})}\mu(A \setminus \{\mathbf{0}\})$.

Write $X_i = \left(X_i^{(1)}, \cdots, X_i^{(d)}\right)$. Assume Var $\left(X_1^{(1)}\right) = \kappa \in (0, \infty)$. For any $X_0 \in \mathbb{R}^d$, we define the random walk $S^{\mu} = \{S_n^{\mu}\}_{n\geq 0}$ on $\mathbb{R}^d$, $d \geq 2$ with step distribution μ starting at X_0 by

$$S_n^{\mu} = \sum_{k=0}^{n} X_k, \quad n \geq 0. \tag{1.1}$$

Let $\{Y_i\}_{i=1}^{\infty}$ be an i.i.d sequence with the common distribution as $X_1^{(1)}$, which is independent of S^{μ}. Define an one-dimensional random walk $R^{\mu} = \{R_n^{\mu}\}_{n\geq 0}$ on $\mathbb{R}$ starting at Y_0 by

$$R_n^{\mu} = \sum_{k=0}^{n} Y_k, \quad n \geq 0.$$

It is well-known that as $\delta \to 0$, rescaled process $\left\{\delta S^{\mu}_{\lfloor \delta^{-2} t \rfloor}\right\}_{t\geq 0}$ converges in law to $\{B(\kappa t)\}_{t\geq 0}$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$ and $B = \{B(t)\}_{t\geq 0}$ is the d-dimensional standard Brownian motion starting at $\mathbf{0}$; and $\left\{\delta R^{\mu}_{\lfloor \delta^{-2} t \rfloor}\right\}_{t\geq 0}$ converges in law to 1-dimensional Brownian motion $\{B(\kappa t)\}_{t\geq 0}$.

To continue, let $D \subset \mathbb{R}^d$ $(d \geq 1)$ be an open simply-connected bounded domain with smooth boundary ∂D and $\mathbf{0} \in D$. For $a, b \in \mathbb{R}$ and $a < 0 < b$, define $\Omega = (a, b) \subset \mathbb{R}$. In the one-dimensional case, we use Ω instead of D to facilitate the distinction between one-dimensional and high-dimensional cases. Denote by $\mathbb{P}^x$ the law of a stochastic process started at x, and $\mathbb{E}^x$ the corresponding expectation. Here "a stochastic process" may be random walks S^{μ} and R^{μ}, and Brownian motion B. Put

$$\tau_D = \inf\{t \geq 0 : \ B(t) \notin D\}.$$

Let $\omega(x, \mathrm{d}z; D)$ be the continuous harmonic measure for $B = \{B(t)\}_{t\geq 0}$ exiting from D when staring at $x \in D$, that is,

$$\omega(x, \mathrm{d}z; D) = \mathbb{P}^x(B(\tau_D) \in \mathrm{d}z). \tag{1.2}$$

For one-dimensional case, let $D = \Omega$ and

$$\omega(x, z; \Omega) = \mathbb{P}^x(B(\tau_\Omega) = z),\ z \in \{a, b\}. \tag{1.3}$$

In fact, (1.3) is also known as a special case of Gambler's ruin probability.

Now we turn to the discrete-time setting. Without loss of generality, in the rest of this paper we will always assume $S_0^\mu = \mathbf{0}$ (resp. $R_0^\mu = 0$), unless otherwise specified. Let

$$T_D = \min\{n \geq 0 :\ \delta S_n^\mu \notin D\} \quad (\text{resp. } T_\Omega = \min\{n \geq 0 :\ \delta R_n^\mu \notin \Omega\}).$$

Define discrete harmonic measure $\omega_\delta(\mathbf{0}, \Gamma; D)$ (resp. $\omega_\delta(0, \Gamma; \Omega)$) for $\{\delta S_n^\mu\}_{n\geq 0}$ (resp. $\{\delta R_n^\mu\}_{n\geq 0}$) exiting from D (resp. Ω) by

$$\omega_\delta(\mathbf{0}, \Gamma; D) = \mathbb{P}\left(\overline{\delta S_{T_D}^\mu} \in \Gamma\right),\ \forall \text{ measurable } \Gamma \subseteq \partial D, \tag{1.4}$$

$$\left(\text{resp. } \omega_\delta(0, z; \Omega) = \mathbb{P}\left(\overline{\delta R_{T_\Omega}^\mu} = z\right),\ \forall z \in \partial\Omega = \{a, b\},\right) \tag{1.5}$$

where $\overline{\delta S_{T_D}^\mu}$ (resp. $\overline{\delta R_{T_\Omega}^\mu}$) is the point on ∂D (resp. $\partial\Omega$) with the smallest distance to $\delta S_{T_D}^\mu$ (resp. $\delta R_{T_\Omega}^\mu$). Note that the choice for $\overline{\delta S_{T_D}^\mu}$ is almost surely unique when δ is sufficiently small.

In statistical physics, there is much theoretical or numerical evidence showing that a number of discrete harmonic measures for random walks (not necessarily Markovian) converge weakly to the corresponding continuous counterparts. Refer to [10, 18, 19, 22, 23, 26] and references therein. Then it is natural to ask how quickly or in what form these discrete harmonic measures converge weakly to the corresponding continuous counterparts. This question originated from the study of harmonic measure error corrections for 2-dimensional random walks [simple random walk (SRW), nearest neighbor random walk not allowed to backtrack (RWNB) and smart kinetic walk (SKW) on square, triangular and hexagonal planar lattices] by Kennedy [23] in 2016. More specifically, there are clear numerical evidences to support the following universality conjecture:

$$\lim_{\delta\to 0} \frac{1}{\delta}\left[\omega_\delta^{M,L}(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D)\right] = C_{M,L}\,\rho_D(\mathbf{0}, z)\,|\mathrm{d}z|, \tag{1.6}$$

where $\omega_\delta^{M,L}(\mathbf{0}, \mathrm{d}z; D)$ is the discrete harmonic measure averaged over orientations (rotations) for the random walks on planar lattices. The reason for averaging over the orientation of the lattice is that Brownian motion is rotationally-invariant and the discrete models are not rotationally-invariant; and in a certain sense, it is natural to take an average over the orientation of the lattice when considering the first-order harmonic measure correction universality. $\rho_D(\mathbf{0}, \cdot)$ is a universal measurable function on ∂D independent of the random walks and lattice (this indicate that there is a sort of universality for first-order correction), and $C_{M,L}$ is a constant dependent on models and lattices but not dependent on the domain. For the details, see [23, Conjecture 1].

The conjecture (1.6) was motivated heuristically in [23] and is still open, and the exact value of the $C_{M,L}$ is unknown. As a contrast to the discrete setting for random walks, in the continuous situation, Jiang and Kennedy [19, Proposition 1] proved rigorously the first-order correction universality conjecture for uniform step μ on $\mathbb{B}^2$ with correction constant

$$K = \frac{16}{45\pi} + \frac{8}{\pi}\int_0^{\pi/2} (\sin^2\theta - (\sin^4\theta)/3 - \theta\cos\theta\sin\theta) E^{i\cos\theta}(|\mathrm{Im}(S^{\mu}_{T_{\mathbb{H}}})|)\mathrm{d}\theta,$$

where $E^{i\cos\theta}$ is the conditional expectation of $\{S^{\mu}_n\}_{n\geq 0}$ given $S_0 = i\cos\theta$, $\mathrm{Im}(z)$ is the imaginary part of z and $T_{\mathbb{H}} := \inf\{n \geq 0 : S^{\mu}_n \notin \mathbb{H}\}$. Monte Carlo simulation of this c_μ gives 0.2647664 ± 0.0000026 (note [19] used K instead of c_μ here); then Wang et al. [31] extended Jiang and Kennedy's conclusion to high dimensional first-order correction of discrete harmonic measures as follows: For the random walk S^{μ} with μ rotationally invariant on $\mathbb{B}^d$ $(d \geq 2)$, and $\mu\{\mathbf{0}\} < 1$, in the sense of the weak convergence topology,

$$\lim_{\delta\to 0}\frac{1}{\delta}\left[\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D)\right] = c_\mu \rho_D(\mathbf{0}, z)\,|\mathrm{d}z|, \tag{1.7}$$

where c_μ is a constant depending only on μ and $\rho_D(\mathbf{0}, z)$ is a measurable function on ∂D independent of μ, and $|dz|$ is the Lebesgue measure on ∂D.

Please note that c_μ given by (1.7) in [31] and K given above are both very complicated. There is usually no valid method for calculating the "expectation term" associated with them. From [19, Remark 4], Jiang and Kennedy conjectured that there seems to be a much simpler expression of K (namely c_μ here). From view points of both theoretic analysis and numerical simulations, it is of interest to seek for a much simpler, beautiful and computable expression for c_μ. This is one aim of our paper. In this paper, we obtain such an expression for c_μ with μ being rotationally invariant on $\mathbb{B}^d$ $(d \geq 2)$, which is given by (1.10) and implied by the proofs of Theorems 1.1 and 1.2. The precise calculation of c_μ requires the study of the overshoot of random walk, roughly speaking, the overshoot of random walk is the quantity of a random walk excess over the boundary. More details with respective to overshoot of random walk refer to [1, 12, 16] and Sect. 2.4.

Besides of our theoretic analysis results (Theorems 1.1 and 1.2), another aim of this paper is to understand further the universality for the first-order and higher-order corrections between discrete harmonic measures and their continuous counterparts by a heuristical argument and numerical simulations. The heuristical argument and numerical simulation evidence lead us to believe that the universality described in Conjectures 3.2 and 3.3 is true. To the best of our knowledge, there are no research conclusions or conjectures that take into account the mentioned high-order corrections in the existing references. In fact, Conjectures 3.2 and 3.3 represent generalizations of first-order corrections between discrete harmonic measures and their continuous counterparts, to be more precise, Conjecture 3.2 is for the rotationally invariant step distributions μ and is proved heuristically, and Conjecture 3.3 is for the generalized discrete harmonic measures (e.g. step distributions μ are not necessarily i.i.d, not necessarily rotationally invariant, even the scaling limit of the random walk needs not be a Brownian motion).

Let $T_l := \min\left\{n \geq 0 :\ R^{\mu}_n \geq l\right\}, l \in \mathbb{R}$. Define $h^{\mu}(l)$ on $[0, 1]$ by

$$\begin{aligned} h^{\mu}(l) = &\int_{[l,1]} \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}\left[\frac{(r^2 - l^2)^{(d-1)/2}}{(d-1)r^{d-2}} + {}_2F_1\left(\frac{1}{2}, \frac{3-d}{2}; \frac{3}{2}; \frac{l^2}{r^2}\right)\frac{l^2}{r}\right] \\ &\times \mathrm{d}\nu(r) - \frac{l}{2}\nu([l, 1]), \end{aligned}$$

where $\nu([0,r]) = \nu(r) := \mu(\{w : |w| \le r\}), r \in [0,1]$ and ${}_2F_1(a,b;c;z)$ is the hypergeometric function given by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, $$

and $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. Let

$$ c_\mu = \frac{2}{\kappa} \int_0^1 \left(l + \mathbb{E}^0 [\left| R^\mu_{T_l} - l \right|] \right) h^\mu(l)\, \mathrm{d}l. \tag{1.8} $$

Our theoretical results are stated in detail as follows:

Theorem 1.1 *The first-order harmonic measure correction constant c_μ for random walk $\{\delta S^\mu_n\}_{n\ge 0} \in \mathbb{R}^d, d \ge 2$ is the same as that of random walk for $\{\delta R^\mu_n\}_{n\ge 0} \in \mathbb{R}$. More precisely, for c_μ specified in (1.8), both (1.7) and the following equality hold:*

$$ \lim_{\delta\to 0} \frac{1}{\delta} [\omega_\delta(0,z;\Omega) - \omega(0,z;\Omega)] = c_\mu \rho_\Omega(0,z), \quad z \in \{a,b\}, $$

where $\omega(0,z;\Omega)$ and $\omega_\delta(0,z;\Omega)$ are given respectively in (1.3) and (1.5), and

$$ \rho_\Omega(0,z) = \begin{cases} \frac{-a-b}{(b-a)^2}, & z = a, \\ \frac{a+b}{(b-a)^2}, & z = b. \end{cases} $$

Theorem 1.1 implies that the calculation of the first-order harmonic measure correction constant for a high-dimensional random walk can be solved by transforming it into a one-dimensional random walk. This insight, which we refer to as the dimension reduction idea, is important in our paper.

To continue, let's recall two concepts, nonlattice and strong nonlattice, for $\mathbb{R}$-valued random variables as follows. Let ξ be a random variable taking values in $\mathbb{R}$, and denote its distribution by η. Say ξ is lattice (arithmetic) if $\eta(\{0, \pm a, \pm 2a, \cdots\}) = 1$ for some $a \in (0,\infty)$, and otherwise nonlattice (non-arithmetic). It is known that ξ is nonlattice if and only if

$$ \phi(t) = \int_{\mathbb{R}} e^{\sqrt{-1}\,tx} \eta(\mathrm{d}x) \ne 1, \ t \ne 0. $$

Say ξ is strongly nonlattice if

$$ \liminf_{|t|\to\infty} |1 - \phi(t)| > 0. $$

Theorem 1.2 *Suppose $X_i \in \mathbb{R}, i \in \mathbb{N}$ are symmetric, nonlattice and i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}\left[X_1^2\right] = \kappa \in (0,\infty)$. Let $R_n = X_1 + \cdots + X_n$. Set*

$$ \omega_\delta(0,z,\Omega) := \mathbb{P}\left(\overline{\delta R_{T_\Omega}} = z\right), z \in \{a,b\} $$

for the discrete harmonic measure of δR_n, and $\omega(0,z,\Omega) := \mathbb{P}(B(\tau_\Omega) = z)$ for the continuous harmonic measure. Write $T_l = \min\{n \ge 1 : R_n \ge l\}$, and

$$ c_* = \lim_{l\to+\infty} \mathbb{E}^0\left[R_{T_l}\right] > 0, \qquad \rho^{(n)}_\Omega(0,z) = \begin{cases} (-2)^{n-1} \frac{-a-b}{(b-a)^{n+1}}, & z = a, \\ (-2)^{n-1} \frac{a+b}{(b-a)^{n+1}}, & z = b, \end{cases} \qquad n \in \mathbb{N}. $$

(i) For any $z \in \{a,b\}$,

$$ \lim_{\delta\to 0} \frac{1}{\delta} (\omega_\delta(0,z,\Omega) - \omega(0,z,\Omega)) = c_* \rho^{(1)}_\Omega(0,z). $$

(ii) Assume further $X_i \in \mathbb{R}, i \in \mathbb{N}$ are also strong nonlattice, and $\mathbb{P}(X_1 \leq t) = o\left(e^{r_1 t}\right)$ as $t \to -\infty$ for some constant $r_1 \in (0, \infty)$. Then for some $r \in (0, \infty)$, as $\delta \to 0$,

$$\omega_\delta(0, z; \Omega) = \frac{|a| + b - |z| + c_* \delta + o(e^{-\frac{r}{\delta}})}{|a| + b + 2c_* \delta + o(e^{-\frac{r}{\delta}})}, \quad z \in \{a, b\},$$

which implies that for any $n \in \mathbb{N}$,

$$\lim_{\delta \to 0} \frac{1}{\delta^n} \left(\omega_\delta(0, z, \Omega) - \omega(0, z, \Omega) - \sum_{k=1}^{n-1} c_*^k \rho_\Omega^{(k)}(0, z) \delta^k \right) = c_*^n \rho_\Omega^{(n)}(0, z), \; z \in \{a, b\}. \tag{1.9}$$

Remark 1.3 Let

$$\mathbb{H}^d := \left\{ (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : \; x_d > 0 \right\}, \; T_{\mathbb{H}^d} := \min \left\{ n \geq 0 : \; S_n^\mu \notin \mathbb{H}^d \right\},$$
$$\ell := (0, \cdots, 0, l) \in \mathbb{R}^d.$$

(i) It is worth noting that random walk R_n^μ in Theorem 1.1 have a probability density function for their step distributions. Hence, R_n^μ in Theorem 1.1 can be seen as a special case of R_n in Theorem 1.2. If step distribution of R_n^μ has same law as that of R_n. Theorems 1.1 and 1.2 imply that the first-order correction constant c_μ(i.e. (1.8)) can be expressed exactly as

$$c_\mu = \frac{2}{\kappa} \int_0^1 \left(l + \mathbb{E}^0 [\left| R_{T_l}^\mu - l \right|] \right) h^\mu(l) \, \mathrm{d}l = \lim_{l \to +\infty} \mathbb{E}^\ell \left[\left| \overline{S_{T_{\mathbb{H}^d}}^\mu} - S_{T_{\mathbb{H}^d}}^\mu \right| \right] = \lim_{l \to +\infty} \mathbb{E}^0 \left[R_{T_l}^\mu \right]. \tag{1.10}$$

This remark confirms the conjecture in Kennedy and Jiang [19, Remark 4]: $K = c_\mu$ is exactly given by the much simpler expression,

$$K = \lim_{l \to +\infty} \mathbb{E}^0 \left[R_{T_l}^\mu \right] = \int_0^\infty -\frac{1}{\pi x^2} \log \left(\frac{8 \left(1 - \frac{2 J_1(x)}{x} \right)}{x^2} \right) \mathrm{d}x = 0.264766405 \cdots.$$

Here μ is the uniform distribution on $\mathbb{B}^2$, $J_1(z)$ is the Bessel function of the first kind of order 1. For the calculation process of K, please skip to Corollary 2.12.

(ii) In Theorem 1.2 (ii), if we replace the condition $\mathbb{P}(X_1 \leq t) = o(e^{r_1 t})$ $(t \to -\infty)$ with $\mathbb{E}(|X_1|^k) < \infty$ for some $k \geq 2$, then by [5, Theorem 1], similarly to Theorem 1.2 (ii), we can prove that as $\delta \to 0$,

$$\omega_\delta(0, z; \Omega) = \frac{|a| + b - |z| + c_* \delta + o(\delta^{k-2})}{|a| + b + 2c_* \delta + o(\delta^{k-2})}, \quad z \in \{a, b\}.$$

In this case (1.9) holds for $n \leq k - 1$.

(iii) As a special case, in Theorem 1.2 (ii), if X_1 has density $\frac{1}{2\lambda} \exp\left(-\frac{|x|}{\lambda}\right)$, $x \in \mathbb{R}$ with $\lambda \in (0, \infty)$, then from Proposition 2.16, for any $\delta \in (0, \infty)$,

$$\omega_\delta(0, z; \Omega) = \frac{|a| + b - |z| + \lambda\delta}{|a| + b + 2\lambda\delta}, \quad z \in \{a, b\}.$$

(iv) It is easy to verify that Theorem 1.1 also holds for X_i which is supported in

$$\mathbb{B}_R^d := \{x | x \in \mathbb{R}^d, |x| < R\} \quad (d \geq 2, 0 < R < +\infty)$$

instead of $\mathbb{B}^d$. Indeed, by multiplying X_i by $1/R$, it degenerate into our model. We believe Theorem 1.1 also holds for X_i is supported in $\mathbb{R}^d, d \geq 2$. However, it requires a good definition of a random walk exiting from the boundary of the domain. For example, especially when $|\delta S^{\mu}_{T_D}| = +\infty$, one can uniformly choose a point on the boundary as the hitting point $\overline{\delta S^{\mu}_{T_D}}$. This is important because when δ is sufficiently small, the choice of $\overline{\delta S^{\mu}_{T_D}}$ may not be almost surely unique, but it could be uniquely determined under probabilistic convergence.

The paper is organized as follows. In Sect. 2, we recall firstly some preliminary facts on Green's function, Poisson kernel, harmonic measures, overshoot of random walk and so on; then after giving a series of lemmas on discrete and continuous harmonic measures, we prove Theorems 1.1 and 1.2. In Sect. 3, we suggest a conjecture for high-order error approximation of generalized discrete harmonic measures and prove the conjecture heuristically for the rotationally invariant case. In Sect. 4, some examples of first-order and second-order error simulations for discrete harmonic measures are given. Finally, in Sect. 5, we give our concluding remarks.

2 The Proof of Main Theorems

2.1 Preliminaries

First, we review some facts about the Green's function and Poisson kernel.

Recall of $\Omega = (a, b) \in \mathbb{R}$, for small $\delta > 0$, let

$$\Omega_2 = \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < \delta\}, \quad \Omega_3 = \{z \in \mathbb{R} \setminus \Omega : \mathrm{dist}(z, \partial\Omega) < \delta\}. \tag{2.1}$$

Set $x, y \in \mathbb{R}^d$, the free-space Green's function in $\mathbb{R}^d$, $d \geq 1$ is known as the Newton kernel, which is defined by

$$G(x, y) := \begin{cases} \frac{1}{2\pi} \log(|x - y|), & d = 2; \\ -\frac{\Gamma(d/2-1)}{4\pi^{d/2}} |x - y|^{2-d}, & d \neq 2. \end{cases}$$

Then the Laplace operator of $G(x, y)$ satisfies

$$\Delta G(x, y) = \boldsymbol{\delta}(x - y),$$

where $\boldsymbol{\delta}(x)$ is the Dirac delta function.

Definition 2.1 Given $z, w \in D \subset \mathbb{R}^d$, $d \in \mathbb{N}$ and $t > 0$, let $p_D(t, z, w)$ be the density of $B(t \wedge \tau_D)$ assuming $B(0) = z$, that is

$$p_D(t, z, w) := \lim_{\epsilon \to 0} \frac{\mathbb{P}^z\left(|B(t) - w| \leq \epsilon; t < \tau_D\right)}{V_d \epsilon^d},$$

where V_d is the volume of $\mathbb{B}^d$, i.e. $V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$. Let $p_D(0, z, \cdot)$ be the Dirac delta function at z and we set $p_D(t, z, w) = 0$ if either z or w is not in D.

Green's function for $D \subset \mathbb{R}^d$. The Green's function for the Laplacian with Dirichlet boundary conditions on D or the Green's function for Brownian motion stopped at ∂D, is defined by

$$G_D(z, w) := \frac{1}{2} \int_0^{\infty} p_D(t, z, w)\, \mathrm{d}t, \qquad z, w \in D,$$

where the multiplicative factor $\frac{1}{2}$ is chosen for convenience.

Lemma 2.2 *Given $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, $G_D(z,\cdot)$ is the unique harmonic function on $D \backslash \{z\}$ such that $G_D(z,w) \to 0$ as $w \to \partial D$, and $G_D(z,w)$ can be expressed as*

$$G_D(z,w) = \begin{cases} -\frac{1}{2\pi}\left(\log(|z-w|) - \mathbb{E}^w\left[\log|B(\tau_D)-z|\right]\right), & d=2; \\ \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\left(|z-w|^{2-d} - \mathbb{E}^w\left[|B(\tau_D)-z|^{2-d}\right]\right), & d \neq 2. \end{cases}$$

Proof For $d = 2$, see the Lemma 3.37 in [27]. The cases $d \neq 2$ can be derived by the argument similar to that of case $d = 2$. □

For a fixed $z \in D \subset \mathbb{R}^d$ and $w \in D$, we set an auxiliary function

$$h(z,w) = \begin{cases} \frac{1}{2\pi}\mathbb{E}^w\left[\log|B(\tau_D)-z|\right], & d=2; \\ -\frac{\Gamma(d/2-1)}{4\pi^{d/2}}\mathbb{E}^w\left[|B(\tau_D)-z|^{2-d}\right], & d \neq 2. \end{cases}$$

Then $h(z,\cdot)$ is a harmonic function on D, which is the solution to the Dirichlet problem for the boundary value given by

$$\varphi(w) = \begin{cases} \frac{1}{2\pi}\log(|z-w|), & d=2, \quad w \in \partial D; \\ -\frac{\Gamma(d/2-1)}{4\pi^{d/2}}|z-w|^{2-d}, & d \neq 2, \quad w \in \partial D. \end{cases}$$

Poisson kernel. Given $D \subset \mathbb{R}^d$ with smooth boundary ∂D, if $z \in D$, $w \in \partial D$, $\mathcal{K}_D(x,z)$ is the Poisson kernel in D and may be defined as the derivative of the Green function $G_D(z,w)$ in the direction $\mathbf{n}_w$, i.e.,

$$\mathcal{K}_D(z,w) := \frac{\partial G_D(z,w)}{\partial \mathbf{n}_w},$$

where $\mathbf{n}_w$ is the inward unit normal at $w \in \partial D$.

It is known that for fixed $x \in D$, $\omega(x, \mathrm{d}z; D)$ is absolutely continuous with respect to $|\mathrm{d}z|$, the Lebesgue measure on ∂D. More precisely, for d-dimensional case($d \geq 2$),

$$\omega(x, \mathrm{d}z; D) = \mathcal{K}_D(x,z)\,|\mathrm{d}z|;$$

For one-dimensional case, let $D = \Omega$,

$$\omega(x, z; \Omega) = \mathcal{K}_\Omega(x,z) = \mathbb{P}^x(B(\tau_\Omega) = z), z \in \{a, b\}.$$

Refer to[14, 15, 17, 21, 27] for more backgrounds and details regarding Green function and the Poisson kernel.

For any bounded function g on $\partial\Omega$, consider the following Dirichlet problem:

$$\begin{cases} \Delta f(z) = 0, & z \in \Omega, \\ f(z) = g(z), & z \in \partial\Omega. \end{cases}$$

The unique solution to the equation above can be written as

$$f(z) = \sum_{w \in \{a,b\}} g(w)\,\omega(z,w;\Omega) = \frac{g(b)-g(a)}{b-a}z + \frac{g(a)b-g(b)a}{b-a}. \tag{2.2}$$

Obviously, $f(z)$ is defined for any $z \in \mathbb{R}$. Let υ be the step distribution of $\{R_n^\mu\}_{n\geq 0}$, so υ is a probability measure on $[-1,1]$. The generator Δ_δ for the random walk $\{\delta R_n^\mu\}_{n\geq 0}$ is given by

$$\Delta_\delta H(z) = \int_{[-1,1]} [H(z+\delta w) - H(z)]\,\mathrm{d}\upsilon(w), \tag{2.3}$$

for any bounded measurable function H on $\mathbb{R}$. Consider the following discrete Dirichlet problem:

$$\begin{cases} \Delta_\delta f_\delta(z) = 0, \ z \in \Omega, \\ f_\delta(z) = g(\widetilde{z}), \ z \in \Omega_3; \end{cases} \tag{2.4}$$

where Ω_3 is given by (2.1), and $\widetilde{z} \in \partial\Omega$ satisfies $|\widetilde{z} - z| = \min\{|\zeta - z| : \zeta \in \partial\Omega\}$. Let $\omega_\delta(z, w; \Omega)$ be discrete harmonic measure for $\{\delta R_n^\mu\}_{n\geq 0}$ exiting from w when staring at $z \in \Omega$. It is easy to see that the function f_δ defined by

$$f_\delta(z) = \sum_{w\in\{a,b\}} g(w)\, \omega_\delta(z, w; \Omega) \tag{2.5}$$

is the unique solution to (2.4). The uniqueness follows from the maximum principle.

Recall that μ is rotationally invariant on $\mathbb{B}^d$ $(d \geq 2)$ with $\mu(\{\mathbf{0}\}) = 0$. Then the k-fold convolution υ^k with $k \geq 1$ is absolutely continuous with respect to the 1-dimensional Lebesgue measure. Define the transition probability density for the random walk $\{\delta R_n^\mu\}_{n\geq 0}$:

$$\begin{aligned} p_\delta(0, x, y) &= \boldsymbol{\delta}(x - y); \\ p_\delta(n, x, y) &= \lim_{\epsilon\to 0} \frac{\mathbb{P}^x(|\delta R_n^\mu - y| \leq \epsilon)}{2\epsilon}, \quad n \in \mathbb{N}, \end{aligned} \tag{2.6}$$

Likewise, define the transition probability density for $\{\delta R_n^\mu\}_{n\geq 0}$ killed on exiting from Ω as follows: For any $x \in \Omega$ and $y \in \mathbb{R}$,

$$\begin{aligned} p_{\Omega,\delta}(0, x, y) &= \boldsymbol{\delta}(x - y); \\ p_{\Omega,\delta}(n, x, y) &= \lim_{\epsilon\to 0} \tfrac{\mathbb{P}^x(|\delta R_n^\mu - y|\leq\epsilon, n<T_\Omega)}{2\epsilon}, \quad n \in \mathbb{N}. \end{aligned}$$

Here $p_{\Omega,\delta}(n, x, y)$ does exist by (2.6) for $n \in \mathbb{N}$.

The killed discrete Green function is defined by

$$G_\delta(x, y) = \sum_{n=0}^{\infty} p_{\Omega,\delta}(n, x, y), \quad x, y \in \Omega.$$

2.2 Some Lemmas

An argument similar to [19, Lemma 3] shows that the following Lemma 2.3 holds.

Lemma 2.3 *For any bounded function $g(x)$, $x \in \partial\Omega$, then we obtain*

$$f_\delta(0) - f(0) = \int_{\Omega_2} G_\delta(0, z)\Delta_\delta f(z)\, \mathrm{d}z. \tag{2.7}$$

Define potential kernel $a_\delta(x)$ for the random walk $\{\delta R_n^\mu\}_{n\geq 0}$ by

$$a_\delta(x) = \sum_{n=1}^{\infty} [p_\delta(n, 0, 0) - p_\delta(n, 0, x)], \quad x \in \mathbb{R} \tag{2.8}$$

For convenience, we write $a(x) := a_1(x)$, $p(n, x, y) := p_1(n, x, y)$. From the (2.6) and (2.8), it is easy to verify that $a(x/\delta) = \delta a_\delta(x)$.

Lemma 2.4 *$a(x)$ is well-defined, and there exists a constant C_0 depending on μ such that as $|x| \to \infty$,*

$$a(x) = \frac{|x|}{\kappa} + C_0 + O\left(|x|^{-1}\right),$$

where the constant in big O term only depends on μ.

Proof The lemma follows from the analogous argument of Lemma 2.4 in [31]. □

Lemma 2.5 *For any $x, y \in \Omega$,*

$$G_\Omega(x, y) = -\frac{|x-y|}{2} + \frac{1}{2}\frac{(x-a)(b-y)+(b-x)(y-a)}{b-a}.$$

Proof According to the Lemma 2.2, the proof is trivial. □

To avoid abuse of notation, for a 1-dimensional inward unit normal $\mathbf{n}_x$ at $x \in \partial\Omega$, we can assume:

$$\mathbf{n}_x = \begin{cases} 1, & x = a; \\ -1, & x = b. \end{cases}$$

Corollary 2.6 *If $|a|, b > \delta$, for $l \in [0, \delta]$,*

$$G_\Omega(0, x + l\mathbf{n}_x) = l\mathcal{K}_\Omega(0, x) = \begin{cases} \frac{b}{b-a} l, & x = a; \\ \frac{-a}{b-a} l, & x = b. \end{cases}$$

Proof This corollary follows immediately from Lemma 2.5. □

Lemma 2.7 *Define the following function in (l, δ) with $0 \le l \le \delta$:*

$$h^\mu(l,\delta) := \int_{[l/\delta,1]} \left\{ \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})} \left[\frac{(\delta^2 r^2 - l^2)^{(d-1)/2}}{(d-1)(\delta r)^{d-2}} + {}_2F_1\left(\frac{1}{2}, \frac{3-d}{2}; \frac{3}{2}; \frac{l^2}{\delta^2 r^2}\right)\frac{l^2}{\delta r}\right] - \frac{l}{2} \right\} \times \mathrm{d}\nu(r),$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function and $\nu(r) := \mu(\{w : |w| \le r\}), r \in [0, 1]$. Let $f(x), x \in \mathbb{R}$ be given by (2.2). If $|a|, b > \delta$, for $l \in [0, \delta]$, then

$$\Delta_\delta f(x + l\mathbf{n}_x) = h^\mu(l, \delta)\frac{\partial f(x)}{\partial \mathbf{n}_x}.$$

Proof For $w = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d (d \ge 2)$, we introduce the d-dimensional spherical polar coordinates transform:

$$\begin{cases} x_1 &= r\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\sin(\varphi_{d-1}), \\ x_2 &= r\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\cos(\varphi_{d-1}), \\ \vdots & \qquad \vdots \\ x_{d-1} &= r\sin(\varphi_1)\cos(\varphi_2), \\ x_d &= r\cos(\varphi_1), \end{cases} \tag{2.9}$$

where $0 \le r < \infty,\ 0 \le \varphi_{d-1} \le 2\pi, 0 \le \varphi_i \le \pi,\ 1 \le i \le d-2$. Then the corresponding Jacobian determinant $\mathbf{J}_d(r) := \mathbf{J}_d(\varphi_1, \cdots, \varphi_{d-1}, r)$ satisfies that

$$\mathbf{J}_d(r) = \frac{\partial(x_1, \cdots, x_{d-1}, x_d)}{\partial(\varphi_1, \cdots, \varphi_{d-1}, r)} = r^{d-1}(\sin\varphi_1)^{d-2}(\sin\varphi_2)^{d-3}\cdots\sin\varphi_{d-2}. \tag{2.10}$$

For the convenience of calculation, for any $w \in \mathbb{B}^d$, write $\rho := \rho(r, \varphi_1, \varphi_2, \cdots, \varphi_{d-1})$ for w in the spherical polar coordinates. We rewrite $\mathrm{d}\mu(w)$ as the form of spherical coordinates:

$$\begin{aligned} \mathrm{d}\mu(w) &= \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) = \frac{\Gamma(d/2)}{2\pi^{d/2} r^{d-1}} \mathrm{d}\nu(r)\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{d-1}, \\ &(r, \varphi_1, \cdots, \varphi_{d-1}) \in [0,1] \times [0,\pi)^{d-2} \times [0, 2\pi). \end{aligned} \tag{2.11}$$

Recall the fact that μ is a common rotationally invariant probability measure on $\mathbb{B}^d \in \mathbb{R}^d$, $d \geq 2$ such that $\mu\{\mathbf{0}\} = 0$. Here we only prove the case for $d \geq 3$, since the proof is similar for the case $d = 2$.

Notice (2.11), and recall the definition of $\Delta_\delta f$ in (2.3) and (2.4). In order to simplify the calculation, let $\widetilde{f}(\mathbf{x}) := f(x)$ with $\mathbf{x} = (x, x_2, \cdots, x_d) \in \mathbb{R}^d$, hence we get

$$\mathbf{n_x} = \begin{cases} (1, 0, \cdots, 0), & \mathbf{x} = (a, x_2, \cdots, x_d); \\ (-1, 0, \cdots, 0), & \mathbf{x} = (b, x_2, \cdots, x_d). \end{cases}$$

Therefore,

$$\begin{aligned} \Delta_\delta f(x + l\mathbf{n}_x) &= \int_{\mathbb{B}^d} \left[\widetilde{f}(\mathbf{x} + l\mathbf{n_x} + \delta w) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \mathrm{d}\mu(w) \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left[\widetilde{f}(\mathbf{x} + l\mathbf{n_x} + \delta\rho) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &= \int_{[0, l/\delta)} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left[\widetilde{f}(\mathbf{x} + l\mathbf{n_x} + \delta\rho) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &\quad + \int_{[l/\delta, 1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left[\widetilde{f}(\mathbf{x}+l\mathbf{n_x}+\delta\rho) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &= \int_{[l/\delta, 1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left[\widetilde{f}(\mathbf{x} + l\mathbf{n_x} + \delta\rho) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &= \int_{[l/\delta, 1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi/2 + \arcsin\left(\frac{l}{\delta r}\right)} \left[\widetilde{f}(\mathbf{x} + l\mathbf{n_x} + \delta\rho) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \\ &\quad \times \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &\quad + \int_{[l/\delta, 1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\arccos\left(\frac{l}{\delta r}\right)} \left[\widetilde{f}(\mathbf{x} + l\mathbf{n_x} - \delta\rho) - \widetilde{f}(\mathbf{x} + l\mathbf{n_x})\right] \\ &\quad \times \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &=: I_1(\mathbf{x}, l) + I_2(\mathbf{x}, l). \end{aligned}$$

Notice the fact that $\frac{\partial \widetilde{f}(\mathbf{x})}{\partial \mathbf{n_x}} = \frac{\partial f(x)}{\partial \mathbf{n}_x}$ and $\frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{n_x}^2} = \frac{\partial^2 f(x)}{\partial \mathbf{n}_x^2} = 0$, which implies that the $O(\delta^2)$ vanish in I_1, I_2. A basic calculation shows

$$\begin{aligned} I_1(\mathbf{x}, l) &= \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[l/\delta, 1]} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi/2 + \arcsin\left(\frac{l}{\delta r}\right)} \delta r \cos(\varphi_1) \times \mathbf{J}_d(r)\, \mathrm{d}\mu(r, \varphi_1, \cdots, \varphi_{d-1}) \\ &= \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[l/\delta, 1]} \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})} \frac{(\delta^2 r^2 - l^2)^{(d-1)/2}}{(d-1)(\delta r)^{d-2}}\, \mathrm{d}\nu(r). \end{aligned}$$

Likewise, we have that

$$
\begin{aligned}
I_2(\mathbf{x}, l) &= \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[l/\delta,1]} \int_0^{\pi} \cdots \int_0^{\arccos\left(\frac{l}{\delta r}\right)} -l\mathbf{J}_d(r) \frac{\Gamma(d/2)}{2\pi^{d/2}(\delta r)^{d-1}} \times \mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{d-2} \mathrm{d}\nu(r) \\
&= \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[l/\delta,1]} \int_0^{\arccos\left(\frac{l}{\delta r}\right)} -l \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma(\frac{d-1}{2})} \sin^{d-2}(\varphi_1)\, \mathrm{d}\varphi_1 \mathrm{d}\nu(r) \\
&= \frac{\partial f(x)}{\partial \mathbf{n}_x} \int_{[l/\delta,1]} \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma(\frac{d-1}{2})} {}_2F_1\left(\frac{1}{2}, \frac{3-d}{2}; \frac{3}{2}; \frac{l^2}{\delta^2 r^2}\right) \frac{l^2}{\delta r} - \frac{l}{2}\, \mathrm{d}\nu(r).
\end{aligned}
$$

The last equation holds by a simple computation as that of Lemma 2.8 in [31]. This completes the proof of lemma. □

The following lemma is an estimation of Green's function $G_\delta(0, z)$ near the boundary of the domain $\partial\Omega$, more estimation of discrete Green's functions for discrete-state random walks on different lattices or domains refer to [3, 7, 13, 19, 24, 31].

Lemma 2.8 *Assume $(-\delta, \delta) \subset \Omega$, $x \in \partial\Omega$. Then for any $l \in [0, \delta]$, as $\delta \to 0$,*

$$
\delta^2 G_\delta(0, z) - \frac{2}{\kappa}\mathcal{K}_\Omega(0, x)\left(l + \mathbb{E}^0\left[|\delta R^\mu_{T_l} - l|\right]\right) = O\left(\delta^2\right),
$$

where $z = x + l\mathbf{n}_x \in \Omega_2$ and the big O term depends on Ω and μ.

Proof The proof of the lemma is analogue to that of Lemma 5 and Proposition 2 in [19] and based on some new estimations, such as the potential kernel $a_\delta(x)$ and the solution of one-dimensional discrete Dirichlet problem, and so on.

First, we need to show that there exists a constant $C > 0$ depending on μ but not on δ such that

$$
\left|\delta^2 G_\delta(0, z) - \frac{2}{\kappa} G_\Omega(0, z)\right| \le C\delta
$$

holds uniformly in $z \in \Omega$ with $|z| > \delta$. For $z \in \mathbb{R}$, let $H_\delta(z) = \delta^2 p_{\Omega,\delta}(0, 0, z) - \delta[a(z/\delta) - C_0]$, define

$$
\begin{aligned}
e_\delta(z) &:= \delta^2 G_\delta(0, z) - H_\delta(z) \\
&= \delta^2 \sum_{k=1}^{\infty} p_{\Omega,\delta}(k, 0, z) + \delta[a(z/\delta) - C_0],\ z \in \mathbb{R}.
\end{aligned}
$$

Recall of υ in (2.3), by the Markov property for $\{\delta R^\mu_n\}_{n\ge 0}$, we get

$$
p_{\Omega,\delta}(k, x, y) = \int_{[-1,1]} p_{\Omega,\delta}(k-1, x, y + \delta\xi)\mathrm{d}\upsilon(\xi), \quad x, y \in \Omega,\ k \in \mathbb{N}.
$$

and by the Fubini theorem, for any $z \in \Omega$,

$$
\begin{aligned}
e_\delta(z) &= \delta^2 p_{\Omega,\delta}(1, 0, z) + \delta^2 a_\delta(z) - \delta C_0 + \delta^2 \sum_{k=2}^{\infty} \int_{[-1,1]} p_{\Omega,\delta}(k-1, 0, z + \delta\xi)\mathrm{d}\upsilon(\xi) \\
&= \delta^2 p_{\Omega,\delta}(1, 0, 0) + \delta^2 \int_{[-1,1]} a_\delta(z + \delta\xi)\mathrm{d}\upsilon(\xi) \\
&\quad -\delta C_0 + \delta^2 \int_{[-1,1]} \sum_{k=1}^{\infty} p_{\Omega,\delta}(k, 0, z + \delta\xi)\mathrm{d}\upsilon(\xi)
\end{aligned}
$$

$$= \int_{[-1,1]} e_\delta(z+\delta\xi)\mathrm{d}\upsilon(\xi).$$

According to the definition of $e_\delta(z)$, it can be easily verified that $e_\delta(z) = \frac{|z|}{\kappa} + O\left(\delta^2/|z|\right), z \in \Omega_3$. So we obtain that

$$\begin{cases} \Delta_\delta e_\delta(z) = 0, & z \in \Omega, \\ e_\delta(z) = \frac{|z|}{\kappa} + O\left(\delta^2/|z|\right), & z \in \Omega_3. \end{cases} \tag{2.12}$$

Recall Lemma 2.5, define $\psi(z) := G_\Omega(0,z) + \frac{|z|}{2} = \frac{1}{2}\frac{-a(b-z)+b(z-a)}{b-a}$, $\psi(z)$ can be extended to a harmonic function in domain containing $\overline{\Omega \cup \Omega_3}$. Indeed, $\psi(z)$ is the harmonic function of $z \in \Omega$ satisfying

$$\begin{cases} \Delta\psi(z) = 0, & z \in \Omega, \\ \psi(z) = \frac{1}{2}\frac{-a(b-z)+b(z-a)}{b-a}, & z \in \Omega_3. \end{cases} \tag{2.13}$$

Subtracting $\frac{2}{\kappa}\times$ (2.13) from (2.12), we get

$$\begin{cases} \Delta_\delta\left[e_\delta(z) - \frac{2}{\kappa}\psi(z)\right] = 0, & z \in \Omega, \\ e_\delta(z) - \frac{2}{\kappa}\psi(z) = O\left(\delta^2/|z|\right), & z \in \Omega_3. \end{cases} \tag{2.14}$$

Note that we assumed $(-\delta,\delta) \subset \Omega$, then the maximum principle for Δ_δ implies that

$$e_\delta(z) - \frac{2}{\kappa}\psi(z) = O(\delta), \quad z \in \Omega.$$

Therefore, we finish proof of the first step after a basic calculation.

$$\begin{aligned} \delta^2 G_\delta(0,z) &= e_\delta(z) - \delta\left[a(z/\delta) - C_0\right] \\ &= \left(\frac{2}{\kappa}\psi(z) + O(\delta)\right) - \left(\frac{2}{\kappa}\frac{|z|}{2} + O\left(\delta^2/|z|\right)\right) \\ &= \frac{2}{\kappa}G_\Omega(0,z) + O(\delta), \end{aligned}$$

where three equalities are true if $|z| > \delta$.

The second step, it suffices to prove for $z = x + l\mathbf{n}_x \in \Omega_2$,

$$e_\delta(z) - \frac{2}{\kappa}\psi(z) = \frac{2}{\kappa}\mathcal{K}_\Omega(0,x)\mathbb{E}^0\left[|\delta R^\mu_{T_l} - l|\right] + O(\delta^2). \tag{2.15}$$

where $T_l := \min\left\{n \ge 0 : \delta R^\mu_n \notin (-\infty, l)\right\}$. We write out the $O\left(\delta^2/|z|\right)$ in (2.14). Then $e_\delta(z) - \frac{2}{\kappa}\psi(z)$ satisfies

$$\begin{cases} \Delta_\delta\left[e_\delta(z) - \frac{2}{\kappa}\psi(z)\right] = 0, & z \in \Omega, \\ e_\delta(z) - \frac{2}{\kappa}\psi(z) = -\frac{2}{\kappa}G_\Omega(0,z) + O(\delta^2), & z \in \Omega_3, \end{cases} \tag{2.16}$$

and recall the Corollary 2.6, and $G_\Omega(0,z)$ can be extend to Ω_3. Indeed, for $l \in [0,\delta]$

$$G_\Omega(0, z - l\mathbf{n}_z) = -l\mathcal{K}_\Omega(0,z) = \begin{cases} -\frac{b}{b-a}l, & z = a; \\ \frac{a}{b-a}l, & z = b. \end{cases} \tag{2.17}$$

Let $F_\delta(0,z)$ be the solution of the following discrete Dirichlet problem

$$\begin{cases} \Delta_\delta F_\delta(0,z) = 0, & z \in \Omega; \\ F_\delta(0,z) = l\mathcal{K}_\Omega(0,x), & z = x - l\mathbf{n}_x \in \Omega_3. \end{cases} \tag{2.18}$$

Then (2.15) follows from (2.16), (2.17) and the following claim: for $z = x + l\mathbf{n}_x \in \Omega_2$

$$F_\delta(0, z) = \mathcal{K}_\Omega(0, z)\mathbb{E}^0\left[|\delta R^\mu_{T_l} - l|\right] + O(\delta^2). \tag{2.19}$$

Observe that $\{R^\mu_n\}(n \le T_\Omega)$ is a martingale. According to the property of martingale, it is easy to check that for $z = x + l\mathbf{n}_x \in \Omega_2$, the solution of (2.18) can be written as

$$F_\delta(0, z) = \mathbb{E}^z\left[\mathcal{K}_\Omega\left(0, \overline{\delta R^\mu_{T_\Omega}}\right)\left|\overline{\delta R^\mu_{T_\Omega}} - \delta R^\mu_{T_\Omega}\right|\right]. \tag{2.20}$$

Recall the $\Omega_2 = (a, a+\delta) \cup (b-\delta, b)$, we may assume $z = b + l\mathbf{n}_b \in (b-\delta, b)$, let

$$l^z_b := \mathbb{E}^z\left[|b - \delta R^\mu_{T_\Omega}|\Big|\overline{\delta R^\mu_{T_\Omega}} = b\right], \quad l^z_a := \mathbb{E}^z\left[|a - \delta R^\mu_{T_\Omega}|\Big|\overline{\delta R^\mu_{T_\Omega}} = a\right].$$

More specifically, l^z_b, l^z_a is the average distance of random walk δR^μ_n started from z under the condition of exiting from the boundary point b, a respectively. As a matter of fact, $0 < l, l^z_a, l^z_b < \delta$, then (2.20) can be expressed as

$$\begin{aligned}
F_\delta(0, z) &= \frac{b + |a| + l^z_a - l}{b + |a| + l^z_b + l^z_a} l^z_b \mathcal{K}_\Omega(0, b) + \frac{l^z_b + l}{b + |a| + l^z_b + l^z_a} l^z_a \mathcal{K}_\Omega(0, a) \\
&= (1 + O(\delta)) l^z_b \mathcal{K}_\Omega(0, b) + O(\delta) l^z_a \mathcal{K}_\Omega(0, a) \\
&= l^z_b \mathcal{K}_\Omega(0, b) + O(\delta^2) \\
&= \mathcal{K}_\Omega(0, z)\mathbb{E}^0\left[|\delta R^\mu_{T_l} - l|\right] + O(\delta^2).
\end{aligned}$$

The last equality holds based on the fact that $l^z_b = (1 + O(\delta))\mathbb{E}^0\left[|\delta R^\mu_{T_l} - l|\right]$ for small δ. The claim (2.19) holds for the similar case $z \in (a, a+\delta)$, which completes the proof of lemma. □

Intuitively, Lemma 2.8 tells us the following fact: the estimation of discrete Green's functions near the boundary is related to the Dirichlet problem, and the solution to the Dirichlet problem is related to the distribution of random walks leaving the boundary. This is why the estimation of discrete Green's functions near the boundary is related to the overshoot of random walks.

2.3 Proof of Theorem 1.1

Let f_δ and f be as in (2.5) and (2.2), respectively. By Lemma 2.3, we get that

$$\begin{aligned}
f_\delta(0) - f(0) &= \sum_{z \in \{a,b\}} [\omega_\delta(0, z; \Omega) - \omega(0, z; \Omega)]\, g(z) = \int_{\Omega_2} G_\delta(0, z)\Delta_\delta f(z)\, \mathrm{d}z \\
&= \int_0^\delta G_\delta(0, a + l\mathbf{n}_a)\Delta_\delta f(a + l\mathbf{n}_a)\, \mathrm{d}l + \int_0^\delta G_\delta(0, b + l\mathbf{n}_b)\Delta_\delta f(b + l\mathbf{n}_b)\, \mathrm{d}l.
\end{aligned}$$

Combining with Lemma 2.7 and Lemma 2.8, a straightforward calculation gives that

$$f_\delta(0) - f(0) = c_\mu \delta \sum_{z \in \{a,b\}} \frac{\partial f(z)}{\partial \mathbf{n}_z} \mathcal{K}_\Omega(0, z) + O(\delta^2),$$

where

$$c_\mu = \frac{2}{\kappa} \int_0^1 \left(l + \mathbb{E}^0[\left|R^\mu_{T_l} - l\right|]\right) h^\mu(l)\, \mathrm{d}l,$$

with $h^{\mu}(l) := h^{\mu}(l, 1)$ given in Lemma 2.7. As the analogue argument as that of $d \geq 2$-dimensional case in [31, Lemma 2.12], one can derive that

$$\sum_{z\in\{a,b\}} \frac{\partial f(z)}{\partial \mathbf{n}_z} \mathcal{K}_\Omega(0, z) = \sum_{z\in\{a,b\}} g(z)\rho_\Omega(0, z), \quad \rho_\Omega(0, z) = \begin{cases} \frac{-a-b}{(b-a)^2}, & z = a, \\ \frac{a+b}{(b-a)^2}, & z = b. \end{cases}$$

We might as well assume that $g(x)$ in the above equation is:

$$g(x) = \begin{cases} 1, & x = a, \\ 0, & x = b. \end{cases} \quad \text{or} \quad g(x) = \begin{cases} 0, & x = a, \\ 1, & x = b. \end{cases}$$

Combining the deductions mentioned above, a basic calculation shows

$$\lim_{\delta\to 0} \frac{1}{\delta} [\omega_\delta(0, z; \Omega) - \omega(0, z; \Omega)] = c_\mu \rho_\Omega(0, z), \quad z \in \{a, b\},$$

By comparing (1.8) with equation (1.7) in [31] term by term. Thus we are arrive at the conclusion that c_μ in (1.8) has same value as that of equation (1.7). So far we have completed proving Theorem 1.1. □

2.4 Correction Constants for Some Special Random Walks

If $R_n = \sum_{k=1}^n X_k, n \in \mathbb{N}$ is a 1-dimensional random walk started at 0, where X_i are i.i.d with common distribution $F := F(t) = \mathbb{P}(X_1 \leq t), t \in \mathbb{R}$. If for $t \geq 0$ there exists almost surely $n \in \mathbb{N}$ such that $R_n > t$, then we can define the quantity

$$\hbar(t) := R_{T_t} - t, \quad \text{with} \quad T_t := \min\{n \geq 1 : R_n > t\}, \quad t \geq 0. \tag{2.21}$$

In the context of random walks, $\hbar(t)$ is also known as *overshoot* or the excess over the boundary. But in the theory of renewal processes, $\hbar(t)$ is frequently called the *residual lifetime* or *excess lifetime*, and R_{T_t} is called the *first ladder height*. See the monographs of Asmussen [1], Feller [12] and Gut [16] for a detailed description. There seem to be few exact calculation for $\hbar(t)$ with fixed $t > 0$ in general random walk. But there is a relatively well-studied theory for the special case $t = 0$ and $t \to \infty$. The extensive reading is available at e.g., [4, 8, 11, 28].

Similar to the definition of [1, Section VIII]. Let $\tau_+ = T_0$ and $\tau_- = \inf\{n \geq 1 : R_n \leq 0\}$ be the first ascending, descending ladder epoch respectively, G_+ be the ascending ladder height distribution $G_+(t) = \mathbb{P}(R_{\tau_+} \leq t)$, and G_- be the descending ladder height distribution $G_-(t) = \mathbb{P}(R_{\tau_-} \leq t)$. In fact, G_+, G_- can be obtained by solving Wiener-Hopf factorization identity (e.g. [1, Theorem 3.1]): $F = G_+ + G_- - G_+ * G_-$, where $*$ denotes convolution.

By iterating the definition of τ_+, τ_-, we can define whole sequences $\{\tau_+(n)\}$, $\{\tau_-(n)\}$ of ladder epochs by $\tau_+(1) = \tau_+$, $\tau_-(1) = \tau_-$, and

$$\tau_+(n+1) = \inf\{k > \tau_+(n) : R_k > R_{\tau_+(n)}\}, \quad n \in \mathbb{N};$$

$$\tau_-(n+1) = \inf\{k > \tau_-(n) : R_k \leq R_{\tau_+(n)}\}, \quad n \in \mathbb{N}.$$

Then $\{R_{\tau_+(n)}\}_{n\geq 1}$, $\{R_{\tau_-(n)}\}_{n\geq 1}$ is called the ascending, descending ladder height process, respectively.

We consider the counting process N defined by

$$N(t) := \sum_{n=1}^{\infty} 1_{\{R_{\tau_+(n)} \leq t\}} = \min\{n : R_{\tau_+(n)} > t\}, \quad t \geq 0,$$

and let $G_n(t)$ be the distribution function of $R_{\tau_+(n)}, n \geq 1$. Thus,

$$G_1(t) = G_+(t), \quad G_n(t) = G_+^{n*}(t), \quad n \in \mathbb{N}.$$

G_+^{n*} is the n-fold convolution of G_+ itself. The associated renewal function can be written as

$$U_+(t) := \mathbb{E}(N(t)) = \sum_{n=1}^{\infty} G_n(t). \tag{2.22}$$

The well-known fact that (e.g. [16, Theorem 5.3]),

$$\mathbb{E}(R_{\tau_+(N(t))}) = \mathbb{E}(R_{\tau_+})\mathbb{E}(N(t)) = \mathbb{E}(R_{\tau_+})U_+(t). \tag{2.23}$$

The following lemma gives an asymptotic estimate of $U_+(t)$ as $t \to \infty$.

Lemma 2.9 (Stone [30, Theorem]) *If R_{τ_+} has finite first moment $\mathbb{E}(R_{\tau_+})$ and finite second moment $\mathbb{E}(R^2_{\tau_+})$, if for some $r_1 \geq 1$, $1 - F(t) = o(e^{-r_1 t})$ as $t \to \infty$, and if F is strongly non-lattice, then for some $r > 0$,*

$$U_+(t) = \frac{t}{\mathbb{E}(R_{\tau_+})} + \frac{\mathbb{E}(R^2_{\tau_+})}{2(\mathbb{E}(R_{\tau_+}))^2} + o(e^{-rt}), \quad \textit{as } t \to \infty.$$

For further estimation of $U_+(t)$ refer to e.g. [5, 6, 9]. The following Corollary 2.10 is an immediate result from (2.22), (2.23) and Lemma 2.9.

Corollary 2.10 *If X_1 is a symmetrical random variable and $F(t)$ is strongly nonlattice, if for some $r_1 \geq 1$, $1 - F(t) = o(e^{-r_1 t})$ as $t \to \infty$, then for some $r > 0$*

$$\mathbb{E}[\hbar(t)] = \frac{\mathbb{E}\left[R^2_{T_0}\right]}{2\mathbb{E}\left[R_{T_0}\right]} + o(e^{-rt}), \quad t \to \infty.$$

The following Lemma 2.11 is a simple variation of Lai [25, Theorem 3].

Lemma 2.11 (Lai [25, Theorem 3]) *Suppose $X_1, X_2, \cdots$ is a sequence of i.i.d. $\mathbb{R}$-valued random variables such that F is nonlattice, and $\mathbb{E}[X_1] = 0$, $\mathbb{E}\left[X_1^2\right] = \kappa \in (0, \infty)$. Let*

$$R_n = X_1 + \cdots + X_n, \; n \in \mathbb{N}. \tag{2.24}$$

Then for all $x > 0$,

$$\lim_{t \to \infty} \mathbb{P}[\hbar(t) \leq x] = \frac{1}{\mathbb{E}\left[R_{T_0}\right]} \int_0^x \mathbb{P}\left[R_{T_0} > t\right] \, \mathrm{d}t, \tag{2.25}$$

where $\mathbb{E}\left[R_{T_0}\right] = \frac{1}{\sqrt{2}} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\left[\mathbb{P}(R_n \leq 0) - \frac{1}{2}\right]\right)$. Moreover,

$$c_* := \lim_{t \to +\infty} \mathbb{E}[\hbar(t)] = \frac{\mathbb{E}\left[R^2_{T_0}\right]}{2\mathbb{E}\left[R_{T_0}\right]}.$$

Due to the complexity of calculations for both $\mathbb{E}\left[R_{T_0}\right]$ and $\mathbb{E}\left[R^2_{T_0}\right]$, Siegmund [29] derived an easier method to calculate for the c_* when $\kappa = 1$, as follows.

$$c_* = -\frac{1}{\pi} \int_0^{\infty} t^{-2} \Re \log\left\{2[1 - \phi(t)]/t^2\right\} \, \mathrm{d}t, \tag{2.26}$$

where $\phi(t) = \mathbb{E}[e^{\sqrt{-1}tX_1}]$. $\Re z$ is the real part of the complex number z.

Analogue to the property of stationary occupation measure for renewal process (e.g. [20, Proposition 9.19]). Under the assumptions of Lemmas 2.11, 2.11 implies that the distribution of $\hbar(t)$ is absolutely continuous with respective to Lebesgue measure. Fixing $t > 0$, and putting

$$\widetilde{T}_t := \min\{n \geq 1 : R_n \geq t\}.$$

We note in particular that $R_{\widetilde{T}_t} - t$ has the same distribution as that of $\hbar(t)$ as $t \to \infty$. Hence Corollary 2.10 also holds by replacing T_t in (2.21) with $\widetilde{T}_t$, which will be used in proving Theorem 1.2.

For general case $\kappa \in (0, \infty)$, by suitable modification to the proof of (2.26), we can show the following corollary.

Corollary 2.12 *Under the assumptions stated in Lemma 2.11, we further assume that X_1 is a symmetric random variable with $\mathbb{E}\left[X_1^2\right] = 1$. Given a fixed value of $\kappa \in (0, \infty)$, if we substitute X_i with $\sqrt{\kappa}X_i$ in equation (2.24), we can derive the following result:*

$$c_* = -\frac{1}{\pi}\int_0^\infty t^{-2}\log\left(\frac{2(1-\widetilde{\phi}(t))}{\kappa t^2}\right)\,\mathrm{d}t,$$

where $\widetilde{\phi}(t) = \mathbb{E}\left[e^{\sqrt{-1}t\sqrt{\kappa}X_1}\right]$.

Proof Let $\phi(t) = \mathbb{E}[e^{\sqrt{-1}tX_1})]$, recall of equation (2.26) and the fact $\widetilde{\phi}(t) = \phi(\sqrt{\kappa}t)$, we obtain

$$\begin{aligned} c_* &= \sqrt{\kappa}\left[-\frac{1}{\pi}\int_0^\infty t^{-2}\Re\log\left\{2[1-\phi(t)]/t^2\right\}\,\mathrm{d}t\right] \\ &= -\frac{1}{\pi}\int_0^\infty t^{-2}\log\left(\frac{2(1-\phi(\sqrt{\kappa}t))}{\kappa t^2}\right)\,\mathrm{d}t \\ &= -\frac{1}{\pi}\int_0^\infty t^{-2}\log\left(\frac{2(1-\widetilde{\phi}(t))}{\kappa t^2}\right)\,\mathrm{d}t. \end{aligned}$$

The last second equality holds due to the symmetry of X_1 and integral transformation. The proof is completed. □

The correction constants c_* for two types of random walks (i.e., with uniform step distribution on $\partial\mathbb{B}^d$ or $\mathbb{B}^d$) are of interest to us. According to Theorem 1.1 and Proposition 2.15 below, the random walk S^μ with a uniform step distribution on $\partial\mathbb{B}^{d+2}$ has the same correction constant c_* as that of S^μ with a uniform step distribution on $\mathbb{B}^d$, $d \in \mathbb{N}$.

The following Lemmas 2.13, 2.14 and Proposition 2.15 are to illustrate the close relation between the uniform random variable on the $\mathbb{B}^d$ and the uniform random variable on the $\partial\mathbb{B}^{d+2}$.

Lemma 2.13 *If $X = \left(X^{(1)}, \cdots, X^{(d)}\right)$, $d \geq 2$, is a uniform random variable on $\partial\mathbb{B}^d$, Then the probability density $f(d,x)$ of $X^{(1)}$ satisfies*

$$f(d,x) = \begin{cases} \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\left(1-x^2\right)^{\frac{d-3}{2}}, & x \in (-1,1); \\ 0, & x \in \mathbb{R}\backslash(-1,1). \end{cases}$$

Proof Set $A_{x,x+\Delta x}=\{w=\varrho(\varphi_1,\cdots,\varphi_{d-1},1)\in\partial\mathbb{B}^d,\varphi_1\in(\arccos(x),\arccos(x+\Delta x))\}$. And let

$Area(A_{x,x+\Delta x})$ be the area of $A_{x,x+\Delta x}$, then a basic computation gives

$$Area(A_{x,x+\Delta x})=\int_0^{2\pi}\int_0^{\pi}\cdots\int_{\arccos(x)}^{\arccos(x+\Delta x)}\mathbf{J}_d(1)\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{d-2}.$$

Let ω_d be the area of $\partial\mathbb{B}^d$, i.e. $\omega_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}$. Therefore, for $x\in(-1,1)$

$$\begin{aligned}f(d,x)&=\lim_{\Delta x\to 0}\frac{1}{\omega_d}\frac{Area(A_{x,x+\Delta x})}{\Delta x}\\&=\lim_{\Delta x\to 0}\frac{1}{\omega_d}\frac{1}{\Delta x}\int_{\arccos(x)}^{\arccos(x+\Delta x)}\frac{2\pi^{(d-1)/2}}{\Gamma\left(\frac{d-1}{2}\right)}\sin^{d-2}(\varphi_1)\,\mathrm{d}\varphi_1\\&=\frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\left(1-x^2\right)^{\frac{d-3}{2}}.\end{aligned}$$

The rest of the proof is obvious. □

Lemma 2.14 *If $X=\left(X^{(1)},\cdots,X^{(d)}\right)$, $d\geq 1$ is the uniform random variable on $\mathbb{B}^d$, Then the probability density $f(d,x)$ of $X^{(1)}$ does exist and it satisfies*

$$f(d,x)=\begin{cases}\frac{\Gamma(d/2+1)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}+1\right)}\left(1-x^2\right)^{\frac{d-1}{2}}, & x\in(-1,1);\\ 0, & x\in\mathbb{R}\backslash(-1,1).\end{cases}$$

Proof The proof of lemma is similar to that of Lemma 2.13. □

Proposition 2.15 *If $\widetilde{X}=\left(\widetilde{X}^{(1)},\cdots,\widetilde{X}^{(d)}\right)$ is the uniform random variable on the $\mathbb{B}^d$, $d\in\mathbb{N}$, and $\widehat{X}=\left(\widehat{X}^{(1)},\cdots,\widehat{X}^{(d+2)}\right)$ is the uniform random variable on the $\partial\mathbb{B}^{d+2}$. Then for $d\in\mathbb{N}$, we have*

$$\widetilde{X}^{(1)}\overset{law}{=}\widehat{X}^{(1)}.$$

Proof The proposition follows from Lemmas 2.13 and 2.14. □

A fascinating example for the random walk with bilateral exponential step distribution. Its fascinating aspect lies in the fact that for any positive value of δ, the harmonic measure of this example can be computed exactly, without the need to consider δ approaching 0. See the Proposition 2.16, which provides a detailed explanation of Remark 1.3(iii).

Proposition 2.16 *For $\lambda\in(0,\infty)$, and let random walk $\{R_n\}_{n\geq 1}$ be the bilateral exponential step distribution on $\mathbb{R}$ with density $\frac{1}{2\lambda}\exp\left(-\frac{|x|}{\lambda}\right)$, $x\in\mathbb{R}$. Then for any $\delta\in(0,\infty)$, $c_*=\lambda$ and*

$$\omega_\delta(0,z;\Omega)=\frac{|a|+b-|z|+\lambda\delta}{|a|+b+2\lambda\delta},\quad z\in\{a,b\}.$$

Proof The proposition follows from the lack of memory of the exponential distribution. We may find the argument similar to that of Chapter I in [12]. i.e. the point of the random walk δR_n first entry into $\mathbb{R}\backslash\Omega$ is independent of the epoch of this entry and its overshoot distance $\left|\delta R_{T_\Omega}-\overline{\delta R_{T_\Omega}}\right|$ at both a and b have same density $\frac{1}{\lambda\delta}\exp\left(-\frac{x}{\lambda\delta}\right)$, which implies that

$$\mathbb{E}\left[\left|\delta R_{T_\Omega}-a\right|\,\middle|\,\overline{\delta R_{T_\Omega}}=a\right]=\mathbb{E}\left[\left|\delta R_{T_\Omega}-b\right|\,\middle|\,\overline{\delta R_{T_\Omega}}=b\right]=\int_0^\infty\frac{x}{\lambda\delta}\exp\left(-\frac{x}{\lambda\delta}\right)\mathrm{d}x=\lambda\delta.$$

Table 1 theoretical values for c_μ

d-dimension	Step distribution μ of $\{S_n^\mu\}_{n\geq 1}$	c_μ
$d=1$	The uniform distribution on $(-1,1)$	0.297952276140383
$d=2$	The uniform distribution on $\partial\mathbb{B}^2$	0.349376861547993
$d=2$	The uniform distribution on $\mathbb{B}^2$	0.264766405680596
$d=3$	The uniform distribution on $\partial\mathbb{B}^3$	0.297952276140383
$d=3$	The uniform distribution on $\mathbb{B}^3$	0.240823087230242
$d=4$	The uniform distribution on $\partial\mathbb{B}^4$	0.264766405680596
$d=4$	The uniform distribution on $\mathbb{B}^4$	0.222445055985682
$\forall d\geq 1$	d-dimensional standard normal distribution	0.582597157939010

Due to the fact that $\mathbb{E}(\delta R_{T_\Omega}) = 0$ and $\omega_\delta(0,a,\Omega)+\omega_\delta(0,b,\Omega) = 1$, a basic calculation yields

$$0 = \mathbb{E}(\delta R_{T_\Omega}) = \omega_\delta(0,a,\Omega)\times(a-\lambda\delta)+\omega_\delta(0,b,\Omega)\times(b+\lambda\delta).$$

The rest of the proof is trivial. □

Next, we will provide several correction constants for some random walks. Combining the Corollary 2.12, Proposition 2.15 and the (i) of Remark 1.3, its not difficult to deduce the following decimal approximation results (up to 15 digits), as shown in Table 1. In following table, S_n^μ is transformed into R_n^μ when $d=1$.

2.5 Proof of Theorem 1.2

The proof of Theorem 1.2 (i) is similar to that of Theorem 1.2 (ii), and is much simpler than the latter. So we only verify Theorem 1.2 (ii). To begin, recall Lemma 2.11. Let $\Lambda(l) = (-\infty, l),\ l>0,$

$$T_{\Lambda(l)} = \min\{n:\ R_n\notin\Lambda(l)\},\ l>0,\ \mathcal{T}_a(\delta)=\min\{n:\ \delta R_n\leq a\},\ \mathcal{T}_b(\delta)=\min\{n:\ \delta R_n\geq b\}.$$

Then $\mathcal{T}_a(\delta)$ and $\mathcal{T}_b(\delta)$ are finite almost surely, and $T_\Omega = \mathcal{T}_a(\delta)\wedge\mathcal{T}_b(\delta)$.

We divide our proof in two steps. Firstly, let ξ be a nonnegative random variable whose law is given by the right hand side of (2.25), we need to verify the following properties: As $\delta\to 0$,

$$\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{\mathcal{T}_{b(\delta)}}-b\right|\leq x\,\middle|\,\mathcal{T}_a(\delta)<\mathcal{T}_b(\delta)\right]\to\mathbb{P}[\xi\leq x],\ x\geq 0; \tag{2.27}$$

$$\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{\mathcal{T}_{a(\delta)}}-a\right|\leq x\,\middle|\,\mathcal{T}_b(\delta)<\mathcal{T}_a(\delta)\right]\to\mathbb{P}[\xi\leq x],\ x\geq 0. \tag{2.28}$$

Let $\left\{\delta\widehat{R}_n\right\}_{n\geq 0}$ be another independent random walk starting from 0 which has the same law as $\{\delta R_n\}_{n\geq 0}$. Notice that

$$\{\mathcal{T}_a(\delta)<\mathcal{T}_b(\delta)\} = \{\max\{\delta R_n:\ n\leq\mathcal{T}_a(\delta)\}<b\}.$$

Hence, by the strong Markov property, given $\mathcal{T}_a(\delta)<\mathcal{T}_b(\delta)$,

$$\left\{R_{n+\mathcal{T}_a(\delta)}-R_{\mathcal{T}_a(\delta)}\right\}_{n\geq 0}\text{ is independent of }R_{\mathcal{T}_a(\delta)}\text{ and has the same law as }\left\{\widehat{R}_n\right\}_{n\geq 0}. \tag{2.29}$$

Since for any $\delta > 0$, given $\mathcal{T}_a(\delta) < \mathcal{T}_b(\delta)$,

$$\begin{aligned}\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| &= \frac{1}{\delta}\left|\left(\delta R_{T_{\Lambda(b/\delta)}} - \delta R_{\mathcal{T}_a(\delta)}\right) - \left(b - \delta R_{\mathcal{T}_a(\delta)}\right)\right| \\ &= \left|\left(R_{T_{\Lambda(b/\delta)}} - R_{\mathcal{T}_a(\delta)}\right) - \left(b/\delta - R_{\mathcal{T}_a(\delta)}\right)\right| \\ &\overset{\text{law}}{=} \left|\widehat{R}_{T_{\Lambda\left(b/\delta - R_{\mathcal{T}_a(\delta)}\right)}} - \left(b/\delta - R_{\mathcal{T}_a(\delta)}\right)\right|, \\ &\quad \left\{\widehat{R}_n\right\}_{n\geq 0} \text{ is independent of } R_{\mathcal{T}_a(\delta)} \text{ and } b/\delta - R_{\mathcal{T}_a(\delta)},\end{aligned}$$

and $b/\delta - R_{\mathcal{T}_a(\delta)} \geq b/\delta \to \infty$ $(\delta \to 0)$; by Lemma 2.11 and Corollary 2.12, we have that as $\delta \to 0$,

$$\mathbb{P}\left[\left|\widehat{R}_{T_{\Lambda\left(b/\delta - R_{\mathcal{T}_a(\delta)}\right)}} - \left(b/\delta - R_{\mathcal{T}_a(\delta)}\right)\right| \leq x \,\middle|\, \mathcal{T}_a(\delta) < \mathcal{T}_b(\delta)\right] \to \mathbb{P}[\xi \leq x],\ x \geq 0.$$

Hence,

$$\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| \leq x \,\middle|\, \mathcal{T}_a(\delta) < \mathcal{T}_b(\delta)\right] \to \mathbb{P}[\xi \leq x],\ x \geq 0.$$

Namely (2.27) is true. Similarly, (2.28) holds. Write $A(\delta) := \{\mathcal{T}_a(\delta) < \mathcal{T}_b(\delta)\}$ and $B(\delta) := \{\mathcal{T}_b(\delta) < \mathcal{T}_a(\delta)\}$, and

$$p_a(\delta) := \mathbb{P}(A(\delta)), \quad p_b(\delta) := \mathbb{P}(B(\delta)).$$

Then

$$p_a(\delta) \to \frac{b}{b + |a|},\ p_b(\delta) \to \frac{|a|}{b + |a|},\ \delta \to 0. \tag{2.30}$$

Note that by Lemma 2.11, as $\delta \to 0$,

$$\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| \leq x\right] \to \mathbb{P}[\xi \leq x],\ x \geq 0. \tag{2.31}$$

Since for any $x \geq 0$,

$$\begin{aligned}&\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| \leq x\right] \\ &= \mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| \leq x \,\middle|\, A(\delta)\right] p_a(\delta) + \mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| \leq x \,\middle|\, B(\delta)\right] p_b(\delta),\end{aligned}$$

by (2.27), (2.30) and (2.31), as $\delta \to 0$,

$$\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{\mathcal{T}_b(\delta)} - b\right| \leq x \,\middle|\, B(\delta)\right] = \mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{T_{\Lambda(b/\delta)}} - b\right| \leq x \,\middle|\, B(\delta)\right] \to \mathbb{P}[\xi \leq x]. \tag{2.32}$$

Similarly, as $\delta \to 0$,

$$\mathbb{P}\left[\frac{1}{\delta}\left|\delta R_{\mathcal{T}_a(\delta)} - a\right| \leq x \,\middle|\, A(\delta)\right] \to \mathbb{P}[\xi \leq x],\ x \geq 0. \tag{2.33}$$

Let

$$c_* = \mathbb{E}[\xi] = \lim_{l\to\infty} \mathbb{E}\left[R_{T_{\Lambda(l)}} - l\right] = \frac{\mathbb{E}\left[R_{T_0}^2\right]}{2\mathbb{E}\left[R_{T_0}\right]}.$$

Combining (2.32), (2.33) as well as Corollary 2.10, for some $r_1, r_2 > 0$, we get

$$\mathbb{E}\left[\left|\delta R_{\mathcal{T}_a(\delta)} - a\right| \middle| A(\delta)\right] = c_*\delta + o(e^{-\frac{r_1}{\delta}}), \quad \delta \to 0; \tag{2.34}$$

$$\mathbb{E}\left[\left|\delta R_{\mathcal{T}_b(\delta)} - b\right| \middle| B(\delta)\right] = c_*\delta + o(e^{-\frac{r_2}{\delta}}), \quad \delta \to 0. \tag{2.35}$$

We proceed to the final step of the proof. Since $\{\delta R_n\}_{n \le T_\Omega}$ is a martingale, we have

$$\mathbb{E}[\delta R_{T_\Omega}] = \mathbb{E}[\delta R_0] = 0.$$

Note that

$$\begin{aligned} R_{T_\Omega} &= \left(\delta R_{\mathcal{T}_a(\delta)} - a + a\right)/\delta\, I_{A(\delta)} + \left(\delta R_{\mathcal{T}_b(\delta)} - b + b\right)/\delta\, I_{B(\delta)} \\ &= \left(-\frac{1}{\delta}\left|\delta R_{\mathcal{T}_a(\delta)} - a\right| + a/\delta\right) I_{A(\delta)} + \left(\frac{1}{\delta}\left|\delta R_{\mathcal{T}_b(\delta)} - b\right| + b/\delta\right) I_{B(\delta)}. \end{aligned}$$

By (2.34) and (2.35), let $r = \min\{r_1, r_2\}$, we obtain

$$0 = \mathbb{E}[\delta R_{T_\Omega}] = \mathbb{P}(A(\delta)) \times (a - c_*\delta + o(e^{-\frac{r}{\delta}})) + \mathbb{P}(B(\delta)) \times (b + c_*\delta + o(e^{-\frac{r}{\delta}})),$$

Since $\mathbb{P}[A(\delta)] + \mathbb{P}[B(\delta)] = 1$, we obtain

$$\mathbb{P}[A(\delta)] = \mathbb{P}\left(\overline{\delta R_{T_\Omega}} = a\right) = \frac{b + c_*\delta + o(e^{-\frac{r}{\delta}})}{b - a + 2c_*\delta + o(e^{-\frac{r}{\delta}})}, \tag{2.36}$$

$$\mathbb{P}[B(\delta)] = \mathbb{P}\left(\overline{\delta R_{T_\Omega}} = b\right) = \frac{-a + c_*\delta + o(e^{-\frac{r}{\delta}})}{b - a + 2c_*\delta + o(e^{-\frac{r}{\delta}})}. \tag{2.37}$$

Recall that $\omega_\delta(0, z, \Omega) = \mathbb{P}\left(\overline{\delta R_{T_\Omega}} = z\right), z \in \{a, b\}$ is the discrete harmonic measure of $\{\delta R_n\}_{n \ge 0}$, and $\omega(0, z, \Omega) := \mathbb{P}(B(\tau_\Omega) = z)$ is the harmonic measure for 1-dimensional Brownian motion, and

$$\mathbb{P}(B(\tau_\Omega) = z) = \begin{cases} \frac{b}{b-a}, & z = a, \\ \frac{-a}{b-a}, & z = b. \end{cases} \tag{2.38}$$

Expanding $\omega_\delta(0, z, \Omega) - \omega(0, z, \Omega)$ into a power series at $\delta = 0$. Therefore, for any $n \in \mathbb{N}$,

$$\lim_{\delta \to 0} \frac{1}{\delta^n}\left(\omega_\delta(0, z, \Omega) - \omega(0, z, \Omega) - \sum_{k=1}^{n-1} c_*^k \rho_\Omega^{(k)}(0, z)\delta^k\right) = c_*^n \rho_\Omega^{(n)}(0, z), \ z \in \{a, b\},$$

where

$$\rho_\Omega^{(n)}(0, z) = \begin{cases} (-2)^{n-1}\frac{-a-b}{(b-a)^{n+1}}, & z = a, \\ (-2)^{n-1}\frac{a+b}{(b-a)^{n+1}}, & z = b. \end{cases}$$

The proof is completed. □

3 High-Order Correction

In this section, we propose two high-order conjectures about the correction of discrete harmonic measures: the conjecture for rotationally invariant case and a more general case. A more general case means that there is no requirement for the step distribution of the random walk to be rotationally invariant, or for the step distribution of the random walk to be i.i.d., or even for the random walk to converge to Brownian motion.

3.1 High-Order Correction for Rotationally Invariant Case

In this subsection, we will propose a conjecture regarding the high-order correction of the harmonic function for the rotationally invariant random walk in $\mathbb{R}^d$, $d \geq 2$, and provide a non-rigorous proof for such conjecture.

For $l \in (0, \infty)$ and small δ, define

$$D_{l\delta} = \left\{ z \in \mathbb{R}^d : \mathrm{dist}(z, D) < l\delta \right\}. \tag{3.1}$$

If $l = 1$, write $D_\delta := D_{l\delta}$. Denote by $\omega(\mathbf{0}, \mathrm{d}\zeta; D_\delta)$ the harmonic measure for the d-dimensional standard Brownian motion exiting from D_δ. To facilitate a better understanding of the conjecture that we are about to present, it is necessary to introduce the following proposition.

Proposition 3.1 *Let $g(z)$ be any bounded smooth function on $z \in \partial D$. For $\mathrm{d}\zeta \subset \partial D_\delta$, $\mathrm{d}z \subset \partial D$ with $\zeta = z - \delta \mathbf{n}_z$. For small enough $\delta > 0$, we can write*

$$\omega(\mathbf{0}, \mathrm{d}\zeta; D_\delta) - \omega(\mathbf{0}, \mathrm{d}z; D) = \sum_{n=1}^{\infty} \delta^n \rho_D^{(n)}(\mathbf{0}, z)|\mathrm{d}z|, \tag{3.2}$$

where $\left\{ \rho_D^{(i)}(\mathbf{0}, z), i = 1, 2 \cdots \right\}$ is a class of measurable functions on ∂D. Then the following equations hold:

$$\int_{\partial D} g(z)\rho_D^{(i)}(\mathbf{0}, z)\ |\mathrm{d}z| = \int_{\partial D} \widehat{g}^{(i)}(z)\omega(\mathbf{0}, \mathrm{d}z; D), \quad i = 1, 2, \cdots \tag{3.3}$$

where $\widehat{g}^{(i)}(z)$ satisfy the following Taylor series at $\delta = 0$, i.e.

$$\int_{\partial D_\delta} g(\zeta + \delta \mathbf{n}_z)\omega(z, \mathrm{d}\zeta; D_\delta) - g(z) = \sum_{i=1}^{\infty} \widehat{g}^{(i)}(z)\delta^i.$$

In particular,

$$\widehat{g}^{(1)}(z) = \frac{\partial f(z)}{\partial \mathbf{n}_z}, \quad \rho_D(\mathbf{0}, z) := \rho_D^{(1)}(\mathbf{0}, z) = \frac{\partial h(z)}{\partial \mathbf{n}_z}, \tag{3.4}$$

here $f(z)$ is a harmonic function in D with boundary value given by $g(z)$, $z \in \partial D$ and $h(z)$ is a harmonic function in D with boundary values given by the Poisson kernel $\mathcal{K}_D(\mathbf{0}, z)$, $z \in \partial D$.

Moreover,

$$\int_{\partial D} \rho_D^{(i)}(\mathbf{0}, z)\ |\mathrm{d}z| = 0, \quad i = 1, 2 \cdots.$$

Proof Let f be the solution of the continuous Dirichlet problem

$$\begin{cases} \Delta f(z) = 0, & z \in D, \\ f(z) = g(z), & z \in \partial D. \end{cases}$$

and we define another solution to the Laplace equation in D_δ. Let $\widehat{f}$ solve

$$\begin{cases} \Delta \widehat{f}(\zeta) = 0, & \zeta \in D_\delta, \\ \widehat{f}(\zeta) = g(z), & \zeta = z - \delta \mathbf{n}_z \in \partial D_\delta, \quad z \in \partial D. \end{cases} \tag{3.5}$$

The equation (3.5) implies that $\widehat{f}$ also solves

$$\begin{cases} \Delta \widehat{f}(z) = 0, & z \in D, \\ \widehat{f}(z) = \widehat{g}(z), & z \in \partial D. \end{cases} \tag{3.6}$$

with

$$\widehat{g}(z) = \int_{\partial D_\delta} g(\zeta + \delta \mathbf{n}_z)\omega(z, \mathrm{d}\zeta; D_\delta), \quad z \in \partial D, \quad \zeta = z - \delta \mathbf{n}_z \in \partial D_\delta.$$

Hence, we have

$$\begin{aligned} \widehat{f}(\mathbf{0}) - f(\mathbf{0}) &= \int_{\partial D_\delta} g(\zeta + \delta \mathbf{n}_z)\, \omega(\mathbf{0}, \mathrm{d}\zeta; D_\delta) - \int_{\partial D} g(z)\, \omega(\mathbf{0}, \mathrm{d}z; D) \\ &= \int_{\partial D} \widehat{g}(z)\, \omega(\mathbf{0}, \mathrm{d}z; D) - \int_{\partial D} g(z)\, \omega(\mathbf{0}, \mathrm{d}z; D) \\ &= \int_{\partial D} (\widehat{g}(z) - g(z))\, \omega(\mathbf{0}, \mathrm{d}z; D) \\ &= \int_{\partial D} \delta^i \sum_{i=1}^{\infty} \widehat{g}^{(i)}(z) \omega(\mathbf{0}, \mathrm{d}z; D). \end{aligned} \tag{3.7}$$

The second equality above holds because the solutions for $\widehat{f}(z)$ in equation (3.5) and (3.6) are the same for $z \in D$. The equation (3.7) has another equivalent expression by changing the integral interval. That is

$$\begin{aligned} \widehat{f}(\mathbf{0}) - f(\mathbf{0}) &= \int_{\partial D_\delta} g(\zeta + \delta \mathbf{n}_z)\, \omega(\mathbf{0}, \mathrm{d}\zeta; D_\delta) - \int_{\partial D} g(z)\, \omega(\mathbf{0}, \mathrm{d}z; D) \\ &= \int_{\partial D} g(z) \left[\omega\left(\mathbf{0}, \mathrm{d}(z - \delta \mathbf{n}_z); D_\delta\right) - \omega(\mathbf{0}, \mathrm{d}z; D)\right] \\ &= \int_{\partial D} g(z) \sum_{n=1}^{\infty} \delta^n \rho_D^{(n)}(\mathbf{0}, z) |\mathrm{d}z|. \end{aligned} \tag{3.8}$$

By comparing the both sides of the equality in (3.7) and (3.8) term by term, we see that the equation (3.3) holds.

Recall that $f(z) = g(z)$, $z \in \partial D$ and $\widehat{f}(z) = \widehat{g}(z)$ closely related to $g(z)$, $\zeta = z - \delta \mathbf{n}_z$, $z \in \partial D$. Hence, the Taylor series of $\widehat{g}(z) - g(z)$ satisfies

$$\widehat{g}(z) - g(z) = \frac{\partial f(z)}{\partial \mathbf{n}_z}\delta + O(\delta^2), \quad z \in \partial D.$$

Therefore, the following equation holds.

$$\int_{\partial D} g(z)\rho_D^{(1)}(\mathbf{0}, z)\, |\mathrm{d}z| = \int_{\partial D} \frac{\partial f(z)}{\partial \mathbf{n}_z}\omega(\mathbf{0}, \mathrm{d}z; D).$$

Defined $\rho_D(\mathbf{0}, z) := \rho_D^{(1)}(\mathbf{0}, z) = \frac{\partial h(z)}{\partial \mathbf{n}_z}$. The proof of Lemma 2.12 in [31] implies that

$$\rho_D(\mathbf{0}, z) = \frac{\partial h(z)}{\partial \mathbf{n}_z}.$$

The equations $\int_{\partial D} \rho_D^{(i)}(\mathbf{0}, z)\, |\mathrm{d}z| = 0,\ i = 1, 2 \cdots$ follows only by setting $g(z) \equiv c$ for a certain constant $c \neq 0$. So far, the proof is completed. □

By the Proposition 3.1, one can accurately calculate the universal measurable function $\rho_D^{(n)}(\mathbf{0}, z)$ of higher-order and obtain some properties of $\rho_D^{(n)}(\mathbf{0}, z)$. These results will be used for the higher-order estimates of the discrete measure in Conjecture 3.2 and 3.3.

Based on deep insight and thinking of correction to the one-dimensional discrete harmonic measure in Theorem 1.2 and Proposition 3.1. We have following conjecture.

Conjecture 3.2 *Assume that $D \subset \mathbb{R}^d (d \geq 2)$ is an open simply-connected bounded domain with $\mathbf{0} \in D$ and ∂D is smooth. And μ is a rotationally invariant probability on $\mathbb{B}^d$ and $\mu(\{\mathbf{0}\}) < 1$. Then we conjecture following holds for any finite $n \in \mathbb{N}$.*

$$\lim_{\delta \to 0} \frac{1}{\delta^n} \left(\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D) - \sum_{k=1}^{n-1} (c_\mu \delta)^k \rho_D^{(k)}(\mathbf{0}, z) |\mathrm{d}z| \right) = c_\mu^n \rho_D^{(n)}(\mathbf{0}, z) |\mathrm{d}z|,$$

where c_μ, $\rho_D^{(n)}$ are specified in (1.10) and (3.2) respectively.

The heuristic derivation of Conjecture 3.2. On the one hand, by Theorem 1.1 and Theorem 1.2, it is not difficult to verify that for $\forall z \in \partial D$, the average distance of random walk $\left\{\delta S_n^\mu\right\}_{n\geq 0}$ under the condition of exiting from z is $c_\mu\delta + o(e^{-r/\delta})$ for some $r > 0$ depending on z as $\delta \to 0$, especially,

$$c_\mu = \lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\left[\left|\delta S_{T_D}^\mu - z\right| \middle| \overline{\delta S_{T_D}^\mu} = z\right], \quad a.s.$$

On the other hand, with $\mathrm{d}\zeta = \mathrm{d}(z - \delta\mathbf{n}_z)$, the Proposition 3.1 implies that

$$\omega\left(\mathbf{0}, \mathrm{d}\zeta; D_{c_\mu\delta}\right) - \omega(\mathbf{0}, \mathrm{d}z; D) = \sum_{n=1}^{\infty}(c_\mu\delta)^n\rho_D^{(n)}(\mathbf{0}, z)|\mathrm{d}z|.$$

Combined with the known conclusions: the first-order correction function of $\omega\left(\mathbf{0}, \mathrm{d}\zeta; D_{c_\mu\delta}\right)$ is the same as that of $\omega_\delta(\mathbf{0}, \mathrm{d}z, D)$, (see to (1.7), note: $\rho_D(\mathbf{0}, z) = \rho_D^{(1)}(\mathbf{0}, z)$, $z \in \partial D$).

From the above reasons, we expect that the Conjecture 3.2 holds.

Notice the fact that the Conjecture 3.2 holds for $n = 1$ (refer to [31, Theorem 1.2]) and is a natural generalization of 1-dimension case (see Theorem 1.2(ii)). The Theorem 1.2(ii) and Conjecture 3.2 imply that one approximate the discrete harmonic measure by computing their analogues for a Brownian motion process with stoping boundaries at $a - c_*\delta$, $b + c_*\delta$ and $\partial D_{c_\mu\delta}$ respectively.

3.2 High-Order Correction for a More General Case

In this subsection, we generalize the Conjecture 3.2.

Reviewing the random walks studied in this paper, it can be found that their scaling limits are Brownian motions. Therefore, not surprisingly, their scaling limits of discrete harmonic measure converge to the continuous counterparts. However, on the one hand, there are lots of random walks whose scaling limits are Brownian motions, but their step distribution are not necessarily i.i.d. (e.g. the SRW on hexagonal planar lattices). On the other hand, there are still many random walks whose scaling limits are not Brownian motions, but their discrete harmonic measures converge to the continuous counterparts, these random walks have similar first-order harmonic measure error correction. For example, RWNB, SKW on square, triangular and hexagonal planar lattices and so on. The reader is referred to [10, 22, 23] and the references therein for further details.

Naturally, we present a more general conjecture on high-order approximation of the harmonic measure error. The conjecture is generalized from two aspects: (i)We consider more general random walks in d dimensions $d \geq 2$, whereas in [23], only 2D random walks (i.e. SRW, RWNB and SKW on square, triangular and hexagonal planar lattices). (ii)We consider n-th order correction ($n \geq 1$) in the discrete harmonic measure error correction, whereas in [23], only first-order correction was considered.

When the random walk is not rotationally invariant. Define the general random walk $S_n = \sum_{i=1}^n X_i$, here the support of the X_i is on the $\mathbb{R}^d$ with finite second moment. The X_i here does not have to be independent and identically distributed. Let $\hat{\omega}_\delta(\mathbf{0}, \mathrm{d}z; D)$, $\hat{\omega}_{\delta,\alpha}(\mathbf{0}, \mathrm{d}z; D)$ be the discrete harmonic measure of random walk $\{\delta S_n\}_{n\geq 1}$ and $\{\delta S_{n,\alpha}\}_{n\geq 1}$, respectively. Here $\delta S_{n,\alpha}$ is the image of δS_n under rotation $\alpha \in \mathrm{SO}(d)$, the special orthogonal rotation group of $d \times d$ orthogonal matrices with determinant 1.

Thus it is natural to redefine the discrete harmonic measure $\omega_\delta(\mathbf{0}, \cdot; D)$ by averaging over the orientation: i.e.,

$$\omega_\delta(\mathbf{0}, \mathrm{d}z; D) = \int_{\mathrm{SO}(d)} \hat{\omega}_{\delta,\alpha}(\mathbf{0}, \mathrm{d}z; D)\, \mathrm{d}\widetilde{m}(\alpha),$$

where $\widetilde{m}$ is the normalized Haar measure on $\mathrm{SO}(d)$, and $\hat{\omega}_{\delta,\alpha}(\mathbf{0}, \mathrm{d}z; D)$ is the image of $\hat{\omega}_\delta(\mathbf{0}, \mathrm{d}z; D)$ under rotation $\alpha \in \mathrm{SO}(d)$. Because of this averaging over the orientation of the lattice, $\omega_\delta(\mathbf{0}, \mathrm{d}z; D)$ is a continuous measure on ∂D. More detailed descriptions with respect to averaging over the orientation can be found in [23, 31].

Based on such definition by averaging over the orientation, we have the following more general conjecture.

Conjecture 3.3 *Assume that $D \subset \mathbb{R}^d$ $(d \geq 2)$ is an open simply-connected bounded domain with $\mathbf{0} \in D$ and ∂D is smooth. For any random walk $\{\delta S_n\}_{n\geq 1}$ starting from $\mathbf{0}$ whose discrete harmonic measure converge weakly to the continuous counterpart, and if there exist a positive, finite, absolute constant $c_* \in (0, \infty)$ depending only on the random walk such that for any $z \in \partial D$ with respect to Lebesgue measure, by averaging over the orientation, as $\delta \to 0$,*

$$\int_{\mathrm{SO}(d)} \mathbb{E}\left[\left|\delta S_{T_D,\alpha} - z\right| \,\middle|\, \overline{\delta S_{T_D,\alpha}} = z \right] \mathrm{d}\widetilde{m}(\alpha) = c_*\delta + o(e^{-\frac{r}{\delta}}), \quad a.s. \quad \textit{for some } r > 0;$$

then for any $n \in \mathbb{N}$,

$$\lim_{\delta\to 0} \frac{1}{\delta^n} \left(\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D) - \sum_{k=1}^{n-1} (c_*\delta)^k \rho_D^{(k)}(\mathbf{0}, z)|\mathrm{d}z| \right) = c_*^n \rho_D^{(n)}(\mathbf{0}, z)|\mathrm{d}z|,$$

where $\omega_\delta(\mathbf{0}, \cdot; D)$ is the discrete harmonic measure by averaging over the orientation, $\rho_D^{(n)}$ are defined in (3.2).

Observe that, if Conjecture 3.3 holds, c_* is also given by

$$c_* = \lim_{l\to+\infty} \int_{\mathrm{SO}(d)} \mathbb{E}^\ell \left[\left| \overline{S_{T_{\mathbb{H}^d},\alpha}} - S_{T_{\mathbb{H}^d},\alpha} \right| \right] \mathrm{d}\widetilde{m}(\alpha), \quad \ell = (0, \cdots, 0, l) \in \mathbb{R}^d. \tag{3.9}$$

The interest of this conjecture lies in that it can provide more accurate estimations in the study of a large of discrete harmonic measures and discrete Green's functions.

In the following, we will provide the correction constants of SRW on some classical lattices by averaging over the orientation. Let us illustrate it with several examples.

Example 3.4 **Correction constant c_* of SRW on triangular planar lattice.**

Considering the SRW $S = \{S_n\}_{n\geq 1}$ on triangular planar lattice. More clearly, S with i.i.d random variable $X_i \in \mathbb{R}^2$ such that

$$\mathbb{P}(X_i = (\cos(\alpha), \sin(\alpha))) = \frac{1}{6}, \ \alpha = \frac{k\pi}{3}, k = 0, ..., 5.$$

In fact, according to (3.9), c_* is given exactly by

$$c_* = \frac{1}{2\pi} \int_0^{2\pi} \hbar(\theta)\mathrm{d}\theta, \quad \hbar(\theta) := \lim_{l\to+\infty} \mathbb{E}^0 \left[R_{T_l,\theta} \right], \tag{3.10}$$

where $R_{n,\theta}$ a.s. the nonlattice random walk on $\mathbb{R}$ with step distribution $X_{i,\theta}$ satisfy

$$\mathbb{P}(X_{i,\theta} = \pm\cos(\theta + \frac{k\pi}{3})) = \frac{1}{6}, \quad k = 0, 1, 2, \ \theta \in [0, 2\pi], \quad i = 1, 2, \ldots,$$

Table 2 theoretical values c_* for different random walk in Conjecture 3.3

d-dimension	Random walk $\{S_n\}_{n\geq 1}$	c_*
$d=2$	SRW on triangular planar lattice	0.360153428425501
$d=2$	SRW on $\mathbb{Z}^2$	0.366026584297563
$d=3$	SRW on $\mathbb{Z}^3$	0.307282689984202
$d=4$	SRW on $\mathbb{Z}^4$	0.271695482505523

and $T_l = \min\{n \geq 1 : R_{n,\theta} \geq l\}$. $\hbar(\theta)$ indeed a.s. exist with respect to the Lebesgue measure for $\theta \in [0, 2\pi]$. Define $\Phi(t,\theta) = \mathbb{E}[e^{\sqrt{-1}tX_{1,\theta}}]$. Combining with the Corollary 2.12, (3.10) can be writhen as

$$c_* = \frac{-1}{2\pi^2}\int_0^{2\pi}\int_0^{\infty}\frac{1}{t^2}\log\left(\frac{4(1-\Phi(t,\theta)))}{t^2}\right)\mathrm{d}t\mathrm{d}\theta.$$

Example 3.5 **Correction constant c_* of SRW on $\mathbb{Z}^d$, $d \geq 2$.**

Considering the SRW $S = \{S_n\}_{n\geq 1}$ on $\mathbb{Z}^d (d \geq 2)$. More specifically, S with i.i.d random variable $X_i \in \mathbb{R}^d$ such that $\mathbb{P}(X_i = \pm e_i) = \frac{1}{2d}$, where e_i is the unit vector of the i-axis. In fact, according to (3.9) and d-dimensional spherical polar coordinates transform (2.9), c_* is given exactly by

$$c_* = \frac{1}{\omega_d}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\hbar(\varphi_1,\cdots,\varphi_{d-1})\mathbf{J}_d(1)\,\mathrm{d}\varphi_1,\cdots,\mathrm{d}\varphi_{d-1}, \tag{3.11}$$

where ω_d is the area of $\partial\mathbb{B}^d$ and $\hbar(\varphi_1,\cdots,\varphi_{d-1}) := \hbar(\theta) = \lim_{l\to+\infty}\mathbb{E}^0\left[R_{T_l,\theta}\right]$. In fact, here $R_{n,\theta}$ is a.s. the nonlattice random walk on $\mathbb{R}$ with step distribution $X_{i,\theta}$ such that

$$\begin{cases}\mathbb{P}(X_{i,\theta} = \pm\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\sin(\varphi_{d-1})) = \frac{1}{2d},\\ \mathbb{P}(X_{i,\theta} = \pm\sin(\varphi_1)\cdots\sin(\varphi_{d-2})\cos(\varphi_{d-1})) = \frac{1}{2d},\\ \qquad\vdots\\ \mathbb{P}(X_{i,\theta} = \pm\sin(\varphi_1)\cos(\varphi_2)) = \frac{1}{2d},\\ \mathbb{P}(X_{i,\theta} = \pm\cos(\varphi_1)) = \frac{1}{2d}.\end{cases}$$

Define $\Phi(t,\theta) = \mathbb{E}[e^{\sqrt{-1}tX_{1,\theta}}]$. Combine with the Corollary 2.12, we get

$$c_* = \frac{-\Gamma(d/2)}{2\pi^{d/2+1}}\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{\infty}\frac{1}{t^2}\log\left(\frac{2d(1-\Phi(t,\theta)))}{t^2}\right)\mathbf{J}_d(1)\,\mathrm{d}t\mathrm{d}\varphi_1,\cdots,\mathrm{d}\varphi_{d-1},$$

Hence, it is readily to deduce the following decimal approximation results, see Table 2.

4 The Simulation for First and Second-Order Correction

We give several numerical simulation examples of first or second-order correction for the Conjecture 3.2 and 3.3. We use three examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ to numerically simulate the first and second-order correction by the Monte Carlo method.

For $w = (x_1, x_2, \cdots, x_d) \in \partial\mathbb{B}_R^d$, we replace it with d-dimensional spherical polar coordinates transform in (2.9), it is well known that

$$\mathcal{K}_{\mathbb{B}_R^d}(z, w) = \frac{R^2 - |z|^2}{\omega_d R \left(R^2 + |z|^2 - 2R|z|\cos(\varphi_1)\right)^{d/2}}, \tag{4.1}$$

when $z \in \{0\}^{d-1} \times (0, R)$.

For more details of Poisson kernel for $\mathbb{B}_R^d$ refer to [2, 15].

Proposition 4.1 *Set* $D = \mathbb{B}_R^d$, $z = (0, \cdots, 0, r)$, $r \in [0, R)$ *and let* $w \in \partial\mathbb{B}_R^d$ *and* $\zeta = w - \delta\mathbf{n}_w$. *Define*

$$H(r, \varphi_1, R, \delta) := \frac{\left((R+\delta)^2 - r^2\right)(R+\delta)^{d-2}}{\omega_d \left((R+\delta)^2 + r^2 - 2(R+\delta)r\cos(\varphi_1)\right)^{d/2}},$$

and set

$$H_\delta^{(n)}(r, \varphi_1, R, \delta) := \frac{\partial^n H(r, \varphi_1, R, \delta)}{\partial\delta^n}.$$

Then as $\delta \to 0$*, we have*

$$\omega(z, \mathrm{d}\zeta; D_\delta) - \omega(z, \mathrm{d}w; D) = \sum_{n=1}^{\infty} \frac{\delta^n}{n!} H_\delta^{(n)}(r, \varphi_1, R, 0)\,|\mathrm{d}w|, \quad \zeta \in \partial\mathbb{B}_{R+\delta}^d.$$

Proof According to d-dimensional spherical polar coordinates transform in (2.9), then $|\mathrm{d}\zeta|$, $|\mathrm{d}w|$ can be written as

$$|\mathrm{d}\zeta| = (R+\delta)^{d-1}\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{d-1}, \qquad |\mathrm{d}w| = R^{d-1}\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{d-1}.$$

and the Poisson kernel in (4.1), with the $\zeta = w - \delta\mathbf{n}_w \in \partial\mathbb{B}_{R+\delta}^d$, then we get

$$\begin{aligned}\omega(z, \mathrm{d}\zeta; D_\delta) - \omega(z, \mathrm{d}w; D) &= \mathcal{K}_{\mathbb{B}_{R+\delta}^d}(z, \zeta)\,|\mathrm{d}\zeta| - \mathcal{K}_{\mathbb{B}_R^d}(z, w)\,|\mathrm{d}w| \\ &= \left(H(r, \varphi_1, R, \delta) - H(r, \varphi_1, R, 0)\right)\,\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{d-1} \\ &= \sum_{n=1}^{\infty} \frac{\delta^n}{n!} H_\delta^{(n)}(r, \varphi_1, R, 0)\,\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{d-1}.\end{aligned}$$

The proof is now complete. □

If we set

$$\rho_{\mathbb{B}_R^d}^{(n)}(z, w) := \frac{1}{n!} H_\delta^{(n)}(r, \varphi_1, R, 0), \quad w \in \partial\mathbb{B}_R^d.$$

Considering the need for the simulation of first and second-order corrections later on, we need to calculate $\rho_{\mathbb{B}_R^d}(z, w) := \rho_{\mathbb{B}_R^d}^{(1)}(z, w)$ and $\rho_{\mathbb{B}_R^d}^{(2)}(z, w)$. A basic calculation yields the following result.

$$\rho_{\mathbb{B}_R^d}^{(1)}(z, w) = \frac{\Gamma(d/2)r\left(r^3(2-d) + r^2(d-4)R\cos\varphi_1 + (2+d)rR^2 - dR^3\cos\varphi_1\right)}{2\pi^{d/2}R^2\left(r^2 - 2rR\cos\varphi_1 + R^2\right)^{d/2+1}}. \tag{4.2}$$

$$
\begin{aligned}
\rho^{(2)}_{\mathbb{B}^d_R}(z,w) = -\frac{1}{2\omega_d\left(R^2-2\cos(\varphi_1)Rr+r^2\right)^{d/2+2}}\Big[& R^{d-4}r(6R^4r-5dr^5 \\
&+6r^5+12R^2r^3+d^2r^5+24R^2r^3\cos^2(\varphi_1)-2R^5d\cos(\varphi_1) \\
&-6R^2dr^3-24Rr^4\cos(\varphi_1)-R^2d^2r^3-24R^3r^2\cos(\varphi_1) \\
&+3R^4dr+2R^3dr^2\cos(\varphi_1)-2Rd^2r^4\cos(\varphi_1) \\
&+2R^4dr\cos^2(\varphi_1)-10R^2dr^3\cos^2(\varphi_1)+2R^3d^2r^2\cos(\varphi_1) \\
&-R^4d^2r\cos^2(\varphi_1)+16Rdr^4\cos(\varphi_1)+R^2d^2r^3\cos^2(\varphi_1)\Big].
\end{aligned}
\tag{4.3}
$$

In the following three examples, we redefine $\left\{S_n^{\mu_j}\right\}_{n\geq 0}$ with μ_j instead of μ which is similar to the definition of $\left\{S_n^{\mu}\right\}_{n\geq 0}$ in (1.1).

Example 4.2 Consider random walks $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ $(j=1,2,3)$ starting at $\mathbf{0}=(0,0)$ with μ_1 the uniform distribution on $\mathbb{B}^2$, μ_2 the uniform distribution on $\partial\mathbb{B}^2$, μ_3 the 2-dimensional standard normal distribution, respectively. Let

$$
z_0 = \left(0, \frac{1}{3}\right), \quad D = \left\{\zeta \in \mathbb{R}^2 : |\zeta + z_0| < 1\right\}.
$$

Write $\mathcal{K}_D(\mathbf{0},\zeta)$ and $\mathcal{K}_{\mathbb{B}^2}(z_0,z)$ for the Poisson kernels of D and $\mathbb{B}^2$ respectively. Translation invariance implies

$$
\mathcal{K}_D(\mathbf{0}, z-z_0) = \mathcal{K}_{\mathbb{B}^2}(z_0,z), \quad z \in \partial\mathbb{B}^2.
$$

Introduce 2-dimensional spherical polar coordinates transform:

$$
z = (r\sin(\phi), r\cos(\phi)), \quad \phi \in [0, 2\pi].
$$

From (4.1), we have $\mathcal{K}_D(\mathbf{0}, z-z_0) = \frac{2}{\pi(5-3\cos(\phi))}$. By the equations (4.2), (4.3) and translation invariance, we can derive that

$$\rho_D^{(1)}(\mathbf{0}, z - z_0) = \frac{3(3 - 5\cos(\phi)}{2\pi(5 - 3\cos(\phi))^2}, \quad z \in \partial\mathbb{B}^2.$$

and

$$\rho_D^{(2)}(\mathbf{0}, z - z_0) = \frac{9(\cos(2\phi) - 36\cos(\phi) + 27)}{8\pi(3\cos(\phi) - 5)^3}, \quad z \in \partial\mathbb{B}^2.$$

Without considering the constant product factor, we write $F_D^{(i)}(\vartheta), i = 1, 2$ as the first and second-order correction for difference of the cumulative distribution function(CDF) between discrete harmonic measure and continuous harmonic measure in D, respectively. Noticing the symmetry properties of $\rho_D^{(i)}$, we might as well define for $F_D^{(i)}(\vartheta), i = 1, 2$ as follows:

$$F_D^{(i)}(\vartheta) = \int_{\Gamma(\vartheta)} \rho_D^{(i)}(\mathbf{0}, z - z_0)\, \mathrm{d}\phi = 2\int_0^{\vartheta} \rho_D^{(i)}(\mathbf{0}, z - z_0)\, \mathrm{d}\phi, \quad \vartheta \in [0, \pi].$$

where $\Gamma(\vartheta) = \{z = (x, y) \in \partial D : z - z_0 = (\cos(\theta), \sin(\theta)), \theta \in [0, \vartheta] \cup [2\pi - \vartheta, 2\pi]\}$, $\vartheta \in [0, \pi]$. And we write $F_{\delta,\mu_j}^{(i)}(\vartheta), i = 1, 2; j = 1, 2, 3$ as the corresponding first and second-order simulation differences. Recall of the c_{μ_j} in Table 1, the definition for $F_{\delta,\mu_j}^{(i)}(\vartheta), i = 1, 2$ are as follows:

$$F_{\delta,\mu_j}^{(1)}(\vartheta) = \frac{1}{c_{\mu_j}\delta} \int_{\Gamma(\vartheta)} (\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D))\, \mathrm{d}\phi, \quad \vartheta \in [0, \pi].$$
$$F_{\delta,\mu_j}^{(2)}(\vartheta) = \frac{1}{c_{\mu_j}\delta} \left(F_{\delta,\mu_j}^{(1)}(\vartheta) - F_D^{(1)}(\vartheta)\right), \ \vartheta \in [0, \pi].$$

In this example, we do simulations with $\delta = 0.1$ for $\omega_\delta(\mathbf{0}, \mathrm{d}z; D)$ by the Monte Carlo method. In our simulation, for each random walk $\delta S_{T_D}^{\mu_j}$, $j = 1, 2, 3.$, we generate 3×10^9 samples. For each sample, we run the $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ until it exits the domain D. Finally, we record the exit point of $\overline{\delta S_{T_D}^{\mu_j}}$ on the ∂D. The simulation results and theoretical calculation results are displayed in Figs. 1 and 2.

Example 4.3 Consider random walks $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ starting at $\mathbf{0} = (0, 0)$ with μ_1 the uniform distribution on $\mathbb{B}^2$, μ_2 the uniform distribution on $\partial\mathbb{B}^2$, μ_3 the 2-dimensional standard normal distribution, μ_4 the SRW on square planar lattice and μ_5 the SRW on triangle planar lattice, respectively. Let

$$D = \left\{(x, y) \in \mathbb{R}^2 : -1 < |y| < 2\right\}, \quad \partial_1 := \{(x, y) \in \mathbb{R}^2 : y = -1\},$$
$$\partial_2 := \{(x, y) \in \mathbb{R}^2 : y = 2\}.$$

Since the harmonic measure is conformally invariant, it is not difficult to deduce the Poisson kernel with respect to $D_\delta = \left\{(x, y) \in \mathbb{R}^2 : -1 - \delta < |y| < 2 + \delta\right\}$, i.e.

$$\mathcal{K}_{D_\delta}(\mathbf{0}, z) = \begin{cases} \dfrac{\sin\left(\frac{\pi}{3+2\delta}(1+\delta)\right)}{2(3+2\delta)\left(\cosh\left(\frac{\pi x}{3+2\delta}\right) - \cos\left(\frac{\pi}{3+2\delta}(1+\delta)\right)\right)}, & z = (x, y) \in \partial_1(\delta); \\ \dfrac{\sin\left(\frac{\pi}{3+2\delta}(2+\delta)\right)}{2(3+2\delta)\left(\cosh\left(\frac{\pi x}{3+2\delta}\right) - \cos\left(\frac{\pi}{3+2\delta}(2+\delta)\right)\right)}, & z = (x, y) \in \partial_2(\delta). \end{cases}$$

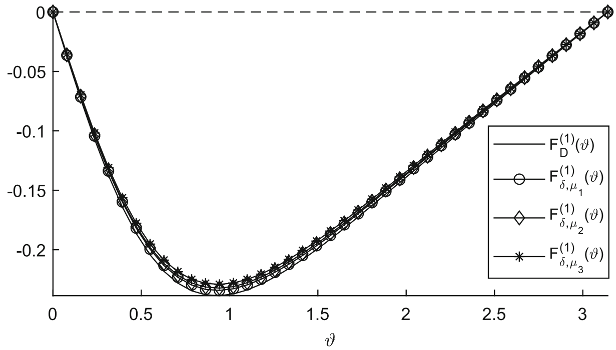

Fig. 1 The first-order rescaled difference $F^{(1)}_{\delta,\mu_j}(\vartheta)$, $i = 1, 2$ from simulations with $\delta = 0.1$

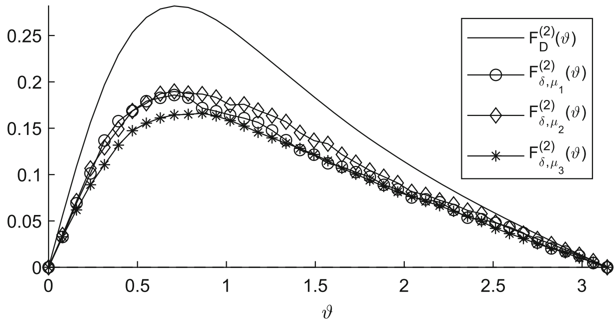

Fig. 2 The second-order rescaled difference $F^{(2)}_{\delta,\mu_j}(\vartheta)$, $i = 1, 2$ from simulations with $\delta = 0.1$

where $\partial_1(\delta) := \{(x, y) \in \mathbb{R}^2 : y = -1 - \delta\}$, $\partial_2(\delta) := \{(x, y) \in \mathbb{R}^2 : y = 2 + \delta\}$. Thence,

$$\mathcal{K}_D(\mathbf{0}, z) = \mathcal{K}_{D_0}(\mathbf{0}, z) = \begin{cases} \frac{1}{\sqrt{3}(4\cosh(\pi x/3)-2)}, & z = (x, y) \in \partial_1; \\ \frac{1}{\sqrt{3}(4\cosh(\pi x/3)+2)}, & z = (x, y) \in \partial_2. \end{cases}$$

By the equation (3.2), we can derive

$$\rho_D^{(1)}(\mathbf{0}, z) = \begin{cases} \frac{2\sqrt{3}\pi x\sinh(\pi x/3)+(\pi-6\sqrt{3})\cosh(\pi x/3)-2\pi+3\sqrt{3}}{27(1-2\cosh(\pi x/3))^2}, & z = (x, y) \in \partial_1; \\ \frac{2\sqrt{3}\pi x\sinh(\pi x/3)+(\pi-6\sqrt{3})\cosh(\pi x/3)+2\pi-3\sqrt{3}}{27(2\cosh(\pi x/3)+1)^2}, & z = (x, y) \in \partial_2. \end{cases}$$

and

$$\rho_D^{(2)}(\mathbf{0}, z) = \begin{cases} \frac{1}{486(2\cosh(\pi x/3)-1)^3}\Big[-12\sqrt{3}\pi^2x^2 + \Big(\sqrt{3}\pi^2(4x^2-1)+120\pi - 144\sqrt{3}\Big)\cosh(\pi x/3) \\ +\Big(\sqrt{3}\pi^2(4x^2-1)-24\pi+72\sqrt{3}\Big)\cosh(2\pi x/3)-28\pi^2x\sinh(\pi x/3)+48\sqrt{3}\pi x\sinh(\pi x/3) \\ +4\pi^2x\sinh(2\pi x/3)-48\sqrt{3}\pi x\sinh(2\pi x/3)+3\sqrt{3}\pi^2-72\pi+108\sqrt{3}\Big], \quad z=(x, y) \in \partial_1; \\ \frac{1}{486(2\cosh(\pi x/3)+1)^3}\Big[-12\sqrt{3}\pi^2x^2+\Big(\sqrt{3}\pi^2(1-4x^2)-120\pi+144\sqrt{3}\Big)\cosh(\pi x/3) \\ +\Big(\sqrt{3}\pi^2(4x^2-1)-24\pi+72\sqrt{3}\Big)\cosh(2\pi x/3)+28\pi^2x\sinh(\pi x/3)-48\sqrt{3}\pi x\sinh(\pi x/3) \\ +4\pi^2x\sinh(2\pi x/3)-48\sqrt{3}\pi x\sinh(2\pi x/3)+3\sqrt{3}\pi^2-72\pi + 108\sqrt{3}\Big], \quad z = (x, y) \in \partial_2. \end{cases}$$

Likewise, similar to the definition in Example 4.2. We use the function $\Gamma(\vartheta)$ of $\vartheta \in [0, \pi]$ to parameterize the boundary of ∂D, more specifically,

$$\Gamma(\vartheta) = \{z = (x, y) \in \partial D : \text{angle}(z, (0, 1)) \leq \vartheta\}, \quad \vartheta \in [0, \pi].$$

where angle$(z, (0, 1))$ means the vector angle between the z and $z' = (0, 1)$. With the correction constants c_{μ_j} in Table 1 and Table 2, define

$$F_D^{(i)}(\vartheta) = \int_{\Gamma(\vartheta)} \rho_D^{(i)}(\mathbf{0}, z)\, \mathrm{d}z, \quad \vartheta \in [0, \pi];$$
$$F_{\delta,\mu_j}^{(1)}(\vartheta) = \frac{1}{c_{\mu_j}\delta} \int_{\Gamma(\vartheta)} (\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D)), \quad \vartheta \in [0, \pi];$$
$$F_{\delta,\mu_j}^{(2)}(\vartheta) = \frac{1}{c_{\mu_j}\delta} \left(F_{\delta,\mu_j}^{(1)}(\vartheta) - F_D^{(1)}(\vartheta)\right), \quad \vartheta \in [0, \pi].$$

In this example, we do simulations with $\delta = 0.1$ and generating 3×10^9 samples for each random walk $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ with $\mu_j = 1, 2, 3$ and with $\delta = 0.02$, 10^8 samples for each random walk $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ with $\mu_j = 4, 5$. The method use here is similar as that of Example 4.2 and the simulation results are shown in Figs. 3 and 4.

Example 4.4 Consider random walks $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ starting at $\mathbf{0} = (0, 0, 0)$ with μ_1 the uniform distribution on $\mathbb{B}^3$, μ_2 the uniform distribution on $\partial\mathbb{B}^3$, μ_3 the 3-dimensional standard normal distribution, μ_4 the SRW on $\mathbb{Z}^3$, respectively. Let

$$z_0 = \left(0, 0, \frac{1}{3}\right), \quad D = \left\{\zeta \in \mathbb{R}^3 : |\zeta + z_0| < 1\right\}.$$

Write $\mathcal{K}_D(\mathbf{0}, \zeta)$ and $\mathcal{K}_{\mathbb{B}^3}(z_0, z)$ for the Poisson kernels of D and $\mathbb{B}^3$ respectively. Obviously,

$$\mathcal{K}_D(\mathbf{0}, z - z_0) = \mathcal{K}_{\mathbb{B}^3}(z_0, z), \quad z \in \partial\mathbb{B}^3.$$

Introduce 3-dimensional spherical polar coordinates transform:

$$z = (r\sin(\phi)\sin(\theta), r\sin(\phi)\cos(\theta), r\cos(\phi)), \quad 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi,\ r \geq 0.$$

From (4.1), $\mathcal{K}_D(\mathbf{0}, z - z_0) = \frac{3}{\sqrt{2}\pi(5-3\cos(\phi))^{3/2}}$.
By the equations (4.2), (4.3) and translation invariance, we can derive that

$$\rho_D^{(1)}(\mathbf{0}, z - z_0) = \frac{33 - 63\cos(\phi)}{4\sqrt{2}\pi(5 - 3\cos(\phi))^{5/2}}, \quad z \in \partial\mathbb{B}^3.$$

and

$$\rho_D^{(2)}(\mathbf{0}, z - z_0) = \frac{27(3\cos^2(\phi) + 20\cos(\phi) - 15)}{8\sqrt{2}\pi(5 - 3\cos(\phi))^{7/2}}, \quad z \in \partial\mathbb{B}^3.$$

A similar reason as that of Example 4.2, define

$$F_D^{(i)}(\vartheta) = \int_0^{2\pi}\int_0^{\vartheta} \rho_D^{(i)}(\mathbf{0}, z - z_0)\sin(\phi)\, \mathrm{d}\phi\mathrm{d}\theta, \quad \vartheta \in [0, \pi];$$
$$F_{\delta,\mu_j}^{(1)}(\vartheta) = \frac{1}{c_{\mu_j}\delta}\int_0^{2\pi}\int_0^{\vartheta} (\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D))\sin(\phi)\, \mathrm{d}\phi\mathrm{d}\theta, \quad \vartheta \in [0, \pi];$$

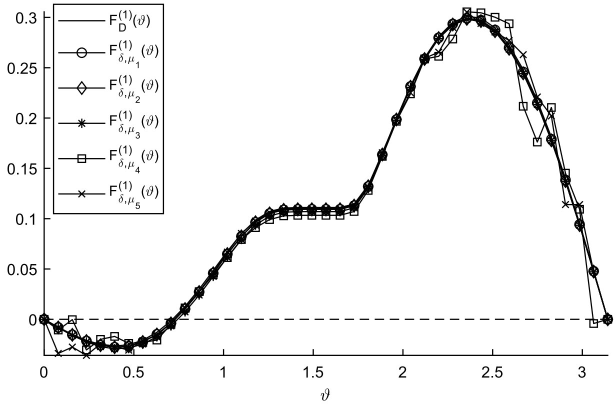

Fig. 3 The first-order rescaled difference $F_{\delta,\mu_j}^{(1)}(\vartheta)$ from simulations in strip domain D

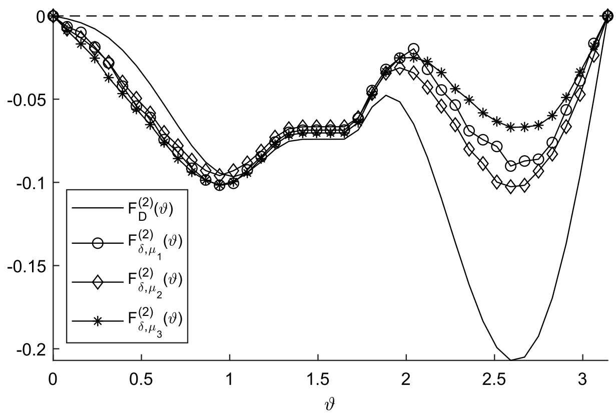

Fig. 4 The second-order rescaled difference $F_{\delta,\mu_j}^{(2)}(\vartheta)$ from simulations with $\delta = 0.1$ in strip domain D

$$F_{\delta,\mu_j}^{(2)}(\vartheta) = \frac{1}{c_{\mu_j}\delta}\left(F_{\delta,\mu_j}^{(1)}(\vartheta) - F_D^{(1)}(\vartheta)\right), \quad \vartheta \in [0, \pi].$$

We do simulations with $\delta = 0.1$ and generating 3×10^9 samples for each random walk $\left\{\delta S_n^{\mu_j}\right\}_{n\geq 0}$ with $\mu_j = 1, 2, 3$ and with $\delta = 0.02$, 2×10^8 samples for random walk $\left\{\delta S_n^{\mu_4}\right\}_{n\geq 0}$. The simulation results are shown in Figs. 5 and 6.

From Figs. 1–6, it seems that the simulation results agree very well with the conjectured counterparts accordingly. Notice that there may be several factor which influence our simulation results, such as the finite number of samples and δ not small enough. In fact, one of the important errors between simulation and theory is that: if the δ is not small enough, there is a small error between the correction constant in the simulation and the theoretical constant c_μ.

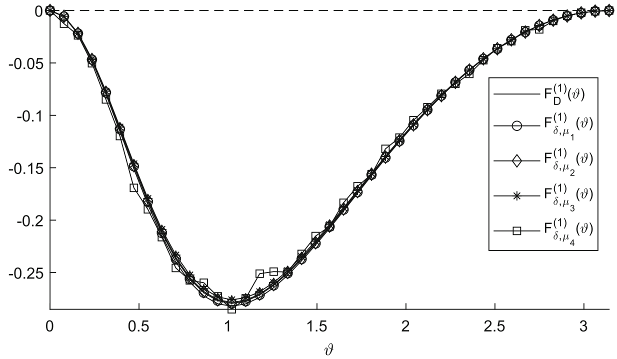

Fig. 5 The first-order rescaled difference $F_{\delta,\mu_j}^{(1)}(\vartheta)$ from simulations

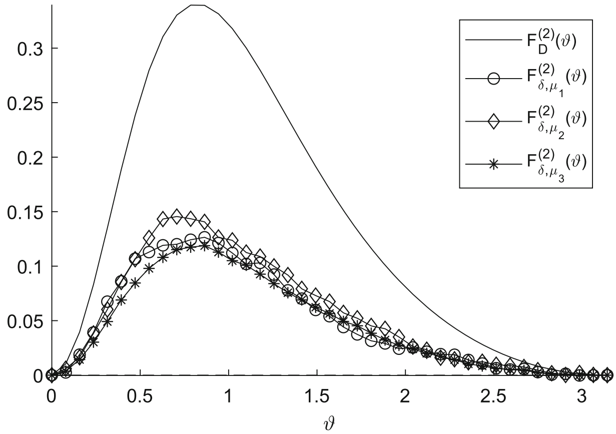

Fig. 6 The second-order rescaled difference $F_{\delta,\mu_j}^{(2)}(\vartheta)$ from simulations with $\delta = 0.1$

5 Concluding Remarks

In this paper, we obtain the following simpler and easily computable expression for the first-order correction constant c_μ between discrete harmonic measures for random walks with rotationally invariant step distribution μ in $\mathbb{R}^d$ $(d \geq 2)$ and the corresponding continuous counterparts (refer to (1.10)):

$$c_\mu = \lim_{l \to +\infty} \mathbb{E}^\ell \left[\left| \overline{S_{T_{\mathbb{H}^d}}^\mu} - S_{T_{\mathbb{H}^d}}^\mu \right| \right], \quad \ell = (0, \cdots, 0, l) \in \mathbb{R}^d.$$

Then the accurate value of c_μ can be calculated through the overshoot of random walk for 1-dimensional random walks. For the non-rotational invariant step distributions μ, we believe c_μ has a similar expression, refer to (3.9). Based on a heuristic deduction and several numerical simulations, we propose a universality conjecture on high-order corrections between generalized discrete harmonic measures and their continuous counterparts in d-dimensional domain $D \subset \mathbb{R}^d$, $d \geq 2$. More clearly, when there is a universality of the first-order correction between discrete harmonic measures and their continuous counterparts, the related high-order corrections must also exist and have the corresponding universality expressions. For example, the random walk with a rotationally invariant step distribution, the SRW, RWNB, SKW, and other random walks having a universality for the first-order corrections, we believe the following expression also holds true for these discrete harmonic measures: for any $n \in \mathbb{N}$,

$$\lim_{\delta \to 0} \frac{1}{\delta^n} \left(\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D) - \sum_{k=1}^{n-1} (c_\mu \delta)^k \rho_D^{(k)}(\mathbf{0}, z)|\mathrm{d}z| \right) = c_\mu^n \rho_D^{(n)}(\mathbf{0}, z)|\mathrm{d}z|.$$

For the details, see Conjectures 3.2 and 3.3.

Furthermore, we have studied numerically the exit distributions of rotational invariant random walks on $\mathbb{R}^d$, $d = 2, 3$, SRW on triangular planar lattice and SRW on $\mathbb{Z}^d$, $d = 2, 3$. All these simulations support the conjecture that the difference between the random walk exit distributions and harmonic measures is, to the first-order and the second-order in the space δ, given by

$$(c_\mu \delta)^i \rho_D^{(i)}(\mathbf{0}, z)\, |\mathrm{d}z|, \ i = 1, 2,$$

where the constant c_μ depends only on the random walks, and the density function $\rho_D^{(i)}(\mathbf{0}, z)$ depends only on the domains. Although we have not provided simulations beyond the third order, but our Conjectures 3.2 and 3.3 suggests that higher-order simulations are also valid. This is because they would require more powerful computers to achieve better simulation results (i.e., smaller delta and more samples). We welcome scholars who are interested in numerical simulations to conduct more in-depth simulations. Thus there is a sort of universalities for these high-order corrections.

Finally, although several numerical simulation examples are given in this paper, it would be interesting to do more test simulations for more random walks. Perhaps the most important question for the future research is to prove Conjecture 3.3 for those classical random walks on lattices. If we weaken the boundary condition of D and could find another effective way to define $\rho_D^{(i)}(\mathbf{0}, z)$, Conjecture 3.3 may hold true for those domains D whose boundaries are piecewise smooth. Refer to [23] for the numerical simulation evidence for the first-order correction of two-dimensional discrete harmonic measures with respect to those D whose boundaries are piecewise smooth. As pointed out by Kennedy [23], there is another very natural way to define the 'exit' point in ∂D when the random walk exits D: By linearly interpolating between the steps of the random walk so that it becomes a piece-wise linear curve in $\mathbb{R}^d$, we can consider the first point where this curve intersects ∂D as the exit point. In this setting, Conjectures 3.2 and 3.3 may hold but with a possibly different correction constant c_μ.

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