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Cutoff for polymer pinning dynamics in the repulsive phase

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Abstract. We consider the Glauber dynamics for model of polymer interacting with a substrate or wall. The state space is the set of one-dimensional nearest-neighbor paths on $\mathbb{Z}$ with nonnegative integer coordinates, starting at 0 and coming back to 0 after L ($L \in 2\mathbb{N}$) steps and the Gibbs weight of a path $\xi = (\xi_x)_{x=0}^{L}$ is given by $\lambda^{\mathcal{N}(\xi)}$, where $\lambda \geq 0$ is a parameter which models the intensity of the interaction with the substrate and $\mathcal{N}(\xi)$ is the number of zeros in ξ. The dynamics proceeds by updating ξ_x with rate one for each $x = 1, \dots, L-1$, in a heat-bath fashion. This model was introduced in (*Electron. J. Probab.* **13** (2008) 213–258) with the aim of studying the relaxation to equilibrium of the system.

We present new results concerning the total variation mixing time for this dynamics when $\lambda < 2$, which corresponds to the phase where the effects of the wall's entropic repulsion dominates. For $\lambda \in [0, 1]$, we prove that the total variation distance to equilibrium drops abruptly from 1 to 0 at time $(L^2 \log L)(1 + o(1))/\pi^2$. For $\lambda \in (1, 2)$, we prove that the system also exhibits cutoff at time $(L^2 \log L)(1 + o(1))/\pi^2$ when considering mixing time from "extremal conditions" (that is, either the highest or lowest initial path for the natural order on the set of paths). Our results improve both previously proved upper and lower bounds in (*Electron. J. Probab.* **13** (2008) 213–258).

Résumé. Nous considérons la dynamique de Glauber pour un modèle de polymère interagissant avec un substrat ou mur. L'espace d'états est l'ensemble des chemins sur $\mathbb{Z}_+$ avec incréments ± 1, commençant en 0 et revenant à 0 après L pas ($L \in 2\mathbb{N}$). Le poids de Gibbs d'un chemin est donné par $\lambda^{\mathcal{N}(\xi)}$, où $\lambda \geq 0$ est un paramètre qui modélise l'intensité de l'interaction avec le substrat et $\mathcal{N}(\xi)$ est le nombre de zéros du chemin ξ. La dynamique procède en mettant à jour ξ_x avec taux un pour chaque $x = 1, \dots, L-1$ à la manière d'un bain de chaleur. Ce modèle a été introduit dans (*Electron. J. Probab.* **13** (2008) 213–258) avec le but d'étudier la relaxation à l'équilibre du système. Nous présentons des nouveaux résultats concernant le temps de mélange de cette dynamique pour la distance en variation totale lorsque $\lambda < 2$. Ce régime correspond à la phase où les effets de répulsion entropique de la paroi dominent. Pour $\lambda \in [0, 1]$, nous prouvons que la distance de variation totale à l'équilibre chute brusquement de 1 à 0 au temps $(L^2 \log L)(1 + o(1))/\pi^2$. Pour $\lambda \in (1, 2)$, nous prouvons que le système présente également un "cutoff" au temps $(L^2 \log L)(1 + o(1))/\pi^2$ en considérant le temps de mélange à partir des conditions extrêmales (c'est-à-dire le chemin initial le plus élevé ou le plus bas pour l'ordre naturel sur l'ensemble des chemins). Nos résultats améliorent les limites supérieures et inférieures déjà prouvées dans (*Electron. J. Probab.* **13** (2008) 213–258).

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Keywords: Polymers; Glauber dynamics; Mixing time; Cutoff

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1. Introduction

1.1. *The random walk pinning model*

Consider the set of all one-dimensional nearest-neighbor paths on $\mathbb{Z}$ with nonnegative integer coordinates, starting at 0 and coming back to 0 after L steps, *i.e.*

$$\Omega_L := \left\{\xi \in \mathbb{Z}^{L+1} : \xi_0 = \xi_L = 0; |\xi_{x+1} - \xi_x| = 1, \forall x \in [\![0, L-1]\!]; \xi_x \geq 0, \forall x \in [\![0, L]\!]\right\},$$

where $L \in 2\mathbb{N}$, and $[\![a, b]\!] := \mathbb{Z} \cap [a, b]$ for all $a, b \in \mathbb{R}$ with $a < b$. We study the polymer pinning model. This model is obtained by assigning to each path $\xi \in \Omega_L$ a weight $\lambda^{\mathcal{N}(\xi)}$, in which $\lambda \geq 0$ is the pinning parameter and

$$\mathcal{N}(\xi) := \sum_{x=1}^{L-1} \mathbf{1}_{\{\xi_x = 0\}} \tag{1.1}$$

is the number of contact points with the x-axis. By convention, $0^0 := 1$ and $0^n := 0$ for any positive integer $n \geq 1$. Normalizing the weights, we obtain a Gibbs probability measure μ_L^λ on Ω_L, defined by

$$\mu_L^\lambda(\xi) := \frac{\lambda^{\mathcal{N}(\xi)}}{Z_L(\lambda)}, \tag{1.2}$$

where $\xi \in \Omega_L$ and

$$Z_L(\lambda) := \sum_{\xi' \in \Omega_L} \lambda^{\mathcal{N}(\xi')}. \tag{1.3}$$

The graph of ξ represents the spatial conformation of the polymer and λ models the energetic interaction with an impenetrable substrate which fills the lower half plane ($\lambda < 1$ corresponding to a repulsive interaction, $\lambda > 1$ to an attractive one). Since $\xi_x \geq 0$ for any $\xi \in \Omega_L$ and any $x \in [\![0, L]\!]$, we say that the polymers interact with an impenetrable substrate. When there is no confusion, we drop the indices λ and L in μ_L^λ.

The random walk pinning model was introduced in the seminal paper [4] several decades ago, and its various derivative models have been studied since. We refer to [5,6] for recent reviews, and mention [5, Chapter 2] and references therein for more details. This model displays a transition from a delocalized phase to a localized phase (see [2, Section 1]): (a) if $0 \leq \lambda < 2$, the expected number of contacts $\mu_L^\lambda(\mathcal{N}(\xi))$ is uniformly bounded in L and the height of the middle point $\xi_{L/2}$ is typically of order $\sqrt{L}$; (b) if $\lambda > 2$, the amount of contacts with the x-axis of typical paths is of order L and the distribution of the height of the middle point $\xi_{L/2}$ is (exponentially) tight in L. These two phases are referred to as the delocalized and localized phase respectively, at the critical point $\lambda = 2$ the system displays an intermediate behavior.

A dynamical version of this model was introduced more recently by Caputo *et al.* in [2]. The corner-flip Glauber dynamics is a continuous-time reversible Markov chain on Ω_L with μ_L^λ as the unique invariant probability measure, whose transitions are given by the updates of local coordinates. We refer to Figure 1 for a graphical description of the jump rates for the system. The dynamics is studied to understand how the system relaxes to equilibrium. Caputo *et al.* in [2, Theorems 3.1 and 3.2] proved that for $\lambda \in [0, 2)$, the mixing time of the dynamics in Ω_L is of order $L^2 \log L$, with non-matching constant prefactors for the upper and lower bounds.

The goal of this paper is to improve both the upper and lower bounds proved in [2] and to show that the mixing time of the system is exactly $(1 + o(1))(L^2 \log L)/\pi^2$ for $\lambda \in [0, 2)$. We prove the result for the worst initial condition mixing time when $\lambda \in [0, 1]$. When $\lambda \in (1, 2)$, our result is valid only for the mixing time starting from either the lowest or highest initial condition but we believe that this is only a technical restriction.

1.2. *The dynamics*

For $\xi \in \Omega_L$ and $x \in [\![1, L-1]\!]$, we define $\xi^x \in \Omega_L$ by

$$\xi_y^x := \begin{cases} \xi_y & \text{if } y \neq x, \\ (\xi_{x-1} + \xi_{x+1}) - \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} \geq 1 \text{ or } \xi_{x-1} \neq \xi_{x+1}, \\ \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} = 0. \end{cases} \tag{1.4}$$

When $\xi_{x-1} = \xi_{x+1}$, ξ displays a local extremum at x and we obtain ξ^x by flipping the corner of ξ at the coordinate x, provided that the path obtained by flipping the corner is in Ω_L (this excludes corner-flipping when $\xi_{x-1} = \xi_{x+1} = 0$). See Figure 1 for a graphical representation. Given the probability measure μ_L^λ defined in (1.2), we construct a continuous-time Markov chain whose generator $\mathcal{L}$ is given by its action on the functions $\mathbb{R}^{\Omega_L}$. It can be written explicitly as

$$(\mathcal{L}f)(\xi) := \sum_{x=1}^{L-1} R_x(\xi)\big[f\big(\xi^x\big) - f(\xi)\big], \tag{1.5}$$

where $f : \Omega_L \to \mathbb{R}$ is a function, and

$$R_x(\xi) := \begin{cases} \frac{1}{2} & \text{if } \xi_{x-1} = \xi_{x+1} > 1, \\ \frac{\lambda}{1+\lambda} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1, 2, 1), \\ \frac{1}{1+\lambda} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1, 0, 1), \\ 0 & \text{if } \xi_{x-1} \neq \xi_{x+1} \text{ or } \xi_{x-1} = \xi_{x+1} = 0. \end{cases}$$

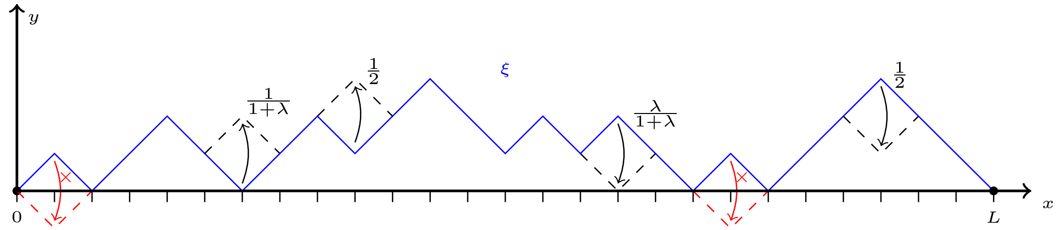

Fig. 1. A graphical representation of the jump rates for the system pinned at $(0,0)$ and $(L,0)$. A transition of the dynamics corresponds to flipping a corner, whose rate depends on how it changes the number of contact points with the x-axis. The rates are chosen in a manner such that the dynamics is reversible with respect to μ_L^λ. The two red dashed corners are not available and labeled with $\times$, because of the nonnegative restriction of the state space Ω_L. Note that not all the possible transitions are shown in the figure.

Equivalently, we can rewrite the generator as

$$(\mathcal{L}f)(\xi) = \sum_{x=1}^{L-1}\big[Q_x(f)(\xi) - f(\xi)\big], \tag{1.6}$$

and

$$Q_x(f)(\xi) := \mu_L^\lambda\big(f \,|\, (\xi_y)_{y\neq x}\big).$$

Let $(\sigma_t^{\xi,\lambda})_{t\geq 0}$ be the trajectory of the Markov chain with initial condition $\sigma_0^{\xi,\lambda} = \xi$ and parameter λ, and let $P_t^{\xi,\lambda}$ be the law of distribution of the time marginal $\sigma_t^{\xi,\lambda}$. Since $\mu_L^\lambda(\xi)R_x(\xi) = \mu_L^\lambda(\xi^x)R_x(\xi^x)$, the continuous-time chain is reversible with respect to the probability measure μ_L^λ. This chain is called the Glauber dynamics. Because the Markov chain is irreducible, by [13, Theorem 3.5.2] we know that for all $\xi \in \Omega_L$, $P_t^{\xi,\lambda}$ converges to μ_L^λ in the discrete topology as t tends to infinity. We ask a quantitative question: how long does it take for $P_t^{\xi,\lambda}$ to converge to μ_L^λ, especially for the worst initial starting path $\xi \in \Omega_L$?

Let us state the aforementioned question in a mathematical framework. If α and β are two probability measures on the space $(\Omega_L, 2^{\Omega_L})$, the total variation distance between α and β is

$$\|\alpha - \beta\|_{\mathrm{TV}} := \frac{1}{2}\sum_{\xi\in\Omega_L}\big|\alpha(\xi) - \beta(\xi)\big| = \sup_{\mathcal{A}\subset\Omega_L}\big(\alpha(\mathcal{A}) - \beta(\mathcal{A})\big). \tag{1.7}$$

We define the distance to equilibrium at time t by

$$d^{L,\lambda}(t) := \max_{\xi\in\Omega_L}\big\|P_t^{\xi,\lambda} - \mu_L^\lambda\big\|_{\mathrm{TV}}. \tag{1.8}$$

For any given $\epsilon \in (0,1)$, let the ϵ-mixing-time be

$$T_{\mathrm{mix}}^{L,\lambda}(\epsilon) := \inf\big\{t \geq 0 : d^{L,\lambda}(t) \leq \epsilon\big\}. \tag{1.9}$$

We say that this sequence of Markov chains has a cutoff, if for all $\epsilon \in (0,1)$,

$$\lim_{L\to\infty}\frac{T_{\mathrm{mix}}^{L,\lambda}(\epsilon)}{T_{\mathrm{mix}}^{L,\lambda}(1-\epsilon)} = 1. \tag{1.10}$$

The cutoff phenomenon is surveyed in the seminal paper [3], and we refer to [12, Chapter 18] for more information. In [2, Theorems 3.1 and 3.2], for all $\lambda \in [0,2)$, Caputo *et al.* proved that for all $\delta > 0$ and all $\epsilon \in (0,1)$, if L is sufficiently large, we have

$$\frac{1-\delta}{2\pi^2}L^2\log L \leq T_{\mathrm{mix}}^{L,\lambda}(\epsilon) \leq \frac{6+\delta}{\pi^2}L^2\log L. \tag{1.11}$$

Moreover, the spectral gap, denoted by $\mathrm{gap}_{L,\lambda}$, is the minimal positive eigenvalue of $-\mathcal{L}$ and the relaxation time $T_{\mathrm{rel}}^{L,\lambda}$ is its inverse. That is

$$T_{\mathrm{rel}}^{L,\lambda} := \sup_{f:\mathrm{Var}_L(f)>0} -\frac{\mathrm{Var}_L(f)}{\mu_L^\lambda(f\mathcal{L}f)} = \mathrm{gap}_{L,\lambda}^{-1}, \tag{1.12}$$

where $\mathrm{Var}_L(f) := \mu_L^\lambda(f^2) - \mu_L^\lambda(f)^2$ with $\mu_L^\lambda(f) := \sum_{\xi\in\Omega_L} \mu_L^\lambda(\xi) f(\xi)$. There is no explicit eigenfunction of the generator $\mathcal{L}$ due to the effect of the impenetrable wall (*i.e.* the x-axis), but Caputo *et al.* adapted the idea in [15, Lemma 1] to find a function (defined in (3.2) below) which is almost an eigenfunction. In [2, Theorems 3.1 and 3.2], they showed that for all $\lambda \in [0, 2)$, there exists a universal constant $C > 0$ independent of L and λ, such that

$$C^{-1}L^2 \le T_{\mathrm{rel}}^{L,\lambda} \le CL^2, \tag{1.13}$$

which together with (1.11) implies $T_{\mathrm{rel}}^{L,\lambda} \ll T_{\mathrm{mix}}^{L,\lambda}(\frac{1}{4})$ and then strongly indicates that Equation (1.10) should hold. Note that the condition $T_{\mathrm{rel}}^{L,\lambda} \ll T_{\mathrm{mix}}^{L,\lambda}(\frac{1}{4})$ is not sufficient to imply the cutoff phenomenon, and we refer to [12, Notes in Chapter 18] for an example.

1.3. *Main results*

In this paper, we find that the mixing time is $(1+o(1))(L^2\log L)/\pi^2$ for all $\lambda \in [0, 1]$, improving both the lower and upper bounds in [2]. That is the following theorem.

Theorem 1.1. *For all* $\epsilon \in (0, 1)$ *and all* $\lambda \in [0, 1]$, *we have*

$$\lim_{L\to\infty} \frac{\pi^2 T_{\mathrm{mix}}^{L,\lambda}(\epsilon)}{L^2 \log L} = 1. \tag{1.14}$$

Therefore, there is a cutoff phenomenon in the Glauber dynamics for $\lambda \in [0, 1]$. The reason why we include the result for $\lambda = 0$ is the need for the mixing time about the dynamics when $\lambda \in (1, 2)$.

Remark 1. Theorem 1.1 about $\lambda = 0$ is the same as the case $\lambda = 1$ by the following identification. Let

$$\Omega_L^+ := \{\xi \in \Omega_L : \mathcal{N}(\xi) = 0\}, \tag{1.15}$$

where $\mathcal{N}(\xi)$ is defined in (1.1), and identify Ω_L^+ with Ω_{L-2} by lifting the x-axis up by distance one in Ω_L. Precisely, the identification is as follows: $\xi = (\xi_x)_{0\le x\le L} \in \Omega_L^+$ is identified with $\varsigma = (\varsigma_x)_{0\le x\le L-2} \in \Omega_{L-2}$, if $\varsigma_x = \xi_{x+1} - 1$ for all $x \in [\![0, L-2]\!]$. We can see:

(a) μ_L^0 is the same as the probability measure μ_{L-2}^1;
(b) the dynamics $(\sigma_t^{\xi,0})_{t\ge0}$ – living in the space Ω_L^+ – is the same as the dynamics $(\sigma_t^{\varsigma,1})_{t\ge0}$ living in the space Ω_{L-2}, where $\xi \in \Omega_L^+$ is identified with $\varsigma \in \Omega_{L-2}$.

Therefore, we only need to prove Theorem 1.1 for $\lambda \in (0, 1]$. In addition, we have a partial result for $\lambda \in (1, 2)$. Let us state the framework. We introduce a natural partial order "$\le$" on Ω_L as follows

$$(\xi \le \xi') \quad \Leftrightarrow \quad (\forall x \in [\![0, L]\!], \xi_x \le \xi'_x).$$

In other words, if $\xi \le \xi'$, the path ξ lies below the path ξ'. Then the maximal path $\wedge$ and the minimal path $\vee$ are respectively given by

$$\wedge_x := \min(x, -x+L), \quad \forall x \in [\![0, L]\!];$$
$$\vee_x := x - 2\lfloor x/2 \rfloor, \quad \forall x \in [\![0, L]\!],$$

where $\lfloor x/2 \rfloor := \sup\{n \in \mathbb{Z} : n \le x/2\}$. Define

$$T_{\mathrm{mix}}^{L,\wedge}(\epsilon) := \inf\{t \ge 0 : \|P_t^{\wedge,\lambda} - \mu_L^\lambda\|_{\mathrm{TV}} \le \epsilon\},$$
$$T_{\mathrm{mix}}^{L,\vee}(\epsilon) := \inf\{t \ge 0 : \|P_t^{\vee,\lambda} - \mu_L^\lambda\|_{\mathrm{TV}} \le \epsilon\},$$

and

$$\check{T}^L_{\text{mix}}(\epsilon) := \max\bigl(T^{L,\wedge}_{\text{mix}}, T^{L,\vee}_{\text{mix}}\bigr). \tag{1.16}$$

For $\lambda \in (1,2)$, applying Peres–Winkler censoring inequality in [14, Theorem 1.1], we discover that the mixing time is also $(1+o(1))(L^2 \log L)/\pi^2$ for the dynamics starting with the two extremal paths. That is the following theorem.

Theorem 1.2. *For all $\epsilon \in (0,1)$ and $\lambda \in (1,2)$, we have*

$$\lim_{L\to\infty} \frac{\pi^2 \check{T}^L_{\text{mix}}(\epsilon)}{L^2 \log L} = 1. \tag{1.17}$$

1.4. *Other values of λ*

Our analysis excludes the case $\lambda > 2$, let us just mention that the convergence to equilibrium follows a different pattern in this case. While the relaxation time and the mixing time are of order L^2 and $L^2 \log L$ respectively in the repulsive phase $\lambda < 2$, it is believed that they become of order L and L^2 respectively in the attractive phase $\lambda > 2$. Rigorous lower bound has been proved in [2, Theorem 3.2], but matching order upper bound has only been shown when $\lambda = \infty$ ([2, Proposition 5.6] for the mixing time). Furthermore in [9, Theorem 2.7], it is shown that in this last case the mixing time is equal to $L^2/4(1+o(1))$. When $\lambda \in (2,\infty)$, the conjecture in [9, Section 2.7] seems to indicate that the mixing time should be of order $C(\lambda)L^2(1+o(1))$ for some explicit $C(\lambda)$.

At the critical value $\lambda = 2$, we believe that the mixing time continues to be $\frac{L^2}{\pi^2}(\log L)(1+o(1))$ but our techniques do not allow to treat this case.

1.5. *Open questions*

Another common distance for measuring how well the Markov chain is mixed is the separation distance, which is defined for any two probability measures α, β on $(\Omega_L, 2^{\Omega_L})$ as

$$d_S(\alpha,\beta) := \max_{\xi\in\Omega_L} \left(1 - \frac{\alpha(\xi)}{\beta(\xi)}\right).$$

Note that it is not a metric for example: taking $\alpha := \frac{1}{2}\delta_\wedge + \frac{1}{2}\delta_\vee$ and $\beta := \frac{1}{3}\delta_\wedge + \frac{2}{3}\delta_\vee$ on the space $(\Omega_L, 2^{\Omega_L})$, and then $d_S(\alpha,\beta) \neq d_S(\beta,\alpha)$ with convention $0/0 = 1$. The separation distance to equilibrium of the dynamics is defined to be

$$s^{L,\lambda}(t) := \max_{\xi\in\Omega_L} d_S\bigl(P^{\xi,\lambda}_t, \mu^\lambda_L\bigr), \tag{1.18}$$

and [12, Lemma 6.16 and Lemma 6.17] tell that

$$d^{L,\lambda}(t) \le s^{L,\lambda}(t) \le 4d^{L,\lambda}(t/2),$$

where the last inequality assumes the reversibility of the dynamics. Moreover, for $\varepsilon \in (0,1)$ we define the ε-separation mixing time as

$$T^{L,\lambda}_{\text{sep}}(\varepsilon) := \inf\bigl\{t \ge 0 : s^{L,\lambda}(t) \le \varepsilon\bigr\}. \tag{1.19}$$

As the heuristic in [15, Section 10.2], we believe that

$$s^{L,\lambda}(t) = 1 - \frac{P^\wedge_t(\vee)}{\mu(\vee)},$$

and the dynamics $(\sigma^\wedge_t)_{t\ge0}$ takes time at most $(1+\delta)\frac{1}{\pi^2}L^2 \log L$ to reach equilibrium in the total variation distance while the dynamics $(\sigma^\vee_t)_{t\ge0}$ takes time of order L^2 to reach equilibrium. Therefore, we expect that for all $\varepsilon \in (0,1)$ and all $\lambda \in [0,2)$

$$\lim_{L\to\infty} \frac{\pi^2 T^{L,\lambda}_{\text{sep}}(\varepsilon)}{L^2 \log L} = 1. \tag{1.20}$$

In Proposition 3.1, we show that for $\lambda \in (0,2)$,

$$\lim_{c\to\infty} \liminf_{L\to\infty} d^{L,\lambda}\left(\frac{1}{\pi^2}L^2\log L - cL^2\right) = 1. \tag{1.21}$$

However, our approach in Proposition 4.2 below is not refined enough to show that for $\lambda \in (0,1]$

$$\lim_{c\to\infty} \limsup_{L\to\infty} d^{L,\lambda}\left(\frac{1}{\pi^2}L^2\log L + cL^2\right) = 0, \tag{1.22}$$

which is believed to be true. A plausible approach is to use the multiscale analysis as in [11, Theorem 1.1] to improve Lemma 4.4 below such that $\mathcal{T}_2 = \frac{1}{\pi^2}L^2\log L + CL^2$ with high probability for C sufficiently large instead of $\mathcal{T}_2 = (1+\frac{\delta}{2})\frac{1}{\pi^2}L^2\log L$. Concerning the problem whether the dynamics mixes in time N^2 starting from a typical initial condition, we believe that the approach in [11] probably works but requires a lot of efforts.

Concerning the cutoff profile, even though Φ defined in (3.2) is good at playing the role of the eigenfunction, the fluctuations of $\Psi(\xi)$, defined in (3.4), at equilibrium are not Gaussian (simply because $\Psi(\xi)$ is positive). For this reason, we believe that the profile is NOT gaussian as in [11].

1.6. *Organization of the paper*

Section 2 introduces a grand coupling for the dynamics corresponding to different ξ and λ, and some useful reclaimed results.

Section 3 is dedicated to the lower bound on the mixing time for $\lambda \in (0,2)$.

Section 4 supplies the upper bound on the mixing time for $\lambda \in (0,1]$.

Section 5 is about the upper bound on the mixing time for the dynamics starting with the two extremal paths when $\lambda \in (1,2)$, applying censoring inequality.

1.7. *Notation*

We use ":=" to define a new quantity on the left-hand side, and use "=:" in some cases when the quantity is defined on the right-hand side.

We let $(C_n(\lambda))_{n\in\mathbb{N}}$ and $(c_n(\lambda))_{n\in\mathbb{N}}$ be some positive constants, which are only dependent on λ. Additionally, we let $(c_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ be some positive and universal constants.

2. Technical preliminaries

To use the monotonicity of the Glauber dynamics, we provide a graphical construction of the Markov chain such that all dynamics, *i.e.* $\{(\sigma_t^{\xi,\lambda})_{t\geq 0} : \forall \xi \in \Omega_L, \forall \lambda \in [0,\infty)\}$, live in one common probability space. This construction appears in [10, Section 8.1], which provides more independent flippable corners than the coupling in [2, Section 2.2.1].

2.1. *A graphical construction*

We set the exponential clocks and independent "coins" in the centers of the squares formed by all the possible corners and their counterparts. Let

$$\Theta := \big\{(x,z) : x \in [\![2, L-2]\!], z \in [\![1, L/2 - 1 - |x - L/2|]\!]; x + z \in 2\mathbb{N}+1\big\}, \tag{2.1}$$

and let $\mathcal{T}^{\uparrow}$ and $\mathcal{T}^{\downarrow}$ be two independent rate-one exponential clock processes indexed by Θ. That is to say, for every $(x,z) \in \Theta$ and $n \geq 0$, we have $\mathcal{T}^{\uparrow}_{(x,z)}(0) = 0$, and

$$\big(\mathcal{T}^{\uparrow}_{(x,z)}(n) - \mathcal{T}^{\uparrow}_{(x,z)}(n-1)\big)_{n\geq 1}$$

is a field of i.i.d. exponential random variables with mean 1. Similarly, this holds for $\mathcal{T}^{\downarrow}_{(x,z)}$. Moreover, let $\mathcal{U}^{\uparrow} = (U^{\uparrow}_{(x,z)}(n))_{(x,z)\in\Theta, n\geq 1}$ and $\mathcal{U}^{\downarrow} = (U^{\downarrow}_{(x,z)}(n))_{(x,z)\in\Theta, n\geq 1}$ be two independent fields of i.i.d. random variables uniformly distributed in $[0,1]$, which are independent of $\mathcal{T}^{\uparrow}$ and $\mathcal{T}^{\downarrow}$. Given $\mathcal{T}^{\uparrow}$, $\mathcal{T}^{\downarrow}$, $\mathcal{U}^{\uparrow}$ and $\mathcal{U}^{\downarrow}$, we construct, in a deterministic way, $(\sigma_t^{\xi,\lambda})_{t\geq 0}$ the trajectory of the Markov chain with parameter λ and starting with $\xi \in \Omega_L$, *i.e.* $\sigma_0^{\xi,\lambda} = \xi$.

When the clock process $\mathcal{T}^{\uparrow}_{(x,z)}$ rings at time $t = \mathcal{T}^{\uparrow}_{(x,z)}(n)$ for $n \geq 1$ and $\sigma_{t^-}^{\xi,\lambda}(x) = z - 1$, we update $\sigma_{t^-}^{\xi,\lambda}$ as follows:

- if $\sigma_{t^-}^{\xi,\lambda}(x-1)=\sigma_{t^-}^{\xi,\lambda}(x+1)=z\geq 2$ and $U_{(x,z)}(n)^{\uparrow}\leq \frac{1}{2}$, let $\sigma_t^{\xi,\lambda}(x)=z+1$ and the other coordinates remain unchanged;
- if $\sigma_{t^-}^{\xi,\lambda}(x-1)=\sigma_{t^-}^{\xi,\lambda}(x+1)=z=1$ and $U_{(x,z)}^{\uparrow}(n)\leq \frac{1}{1+\lambda}$, let $\sigma_t^{\xi,\lambda}(x)=2$ and the other coordinates remain unchanged.

If neither of these two aforementioned conditions is satisfied, we do nothing.

When the clock process $\mathcal{T}_{(x,z)}^{\downarrow}$ rings at time $t=\mathcal{T}_{(x,z)}^{\downarrow}(n)$ for $n\geq 1$ and $\sigma_{t^-}^{\xi,\lambda}(x)=z+1$, we update $\sigma_{t^-}^{\xi,\lambda}$ as follows:

- if $\sigma_{t^-}^{\xi,\lambda}(x-1)=\sigma_{t^-}^{\xi,\lambda}(x+1)=z\geq 2$ and $U_{(x,z)}^{\downarrow}(n)\leq \frac{1}{2}$, let $\sigma_t^{\xi,\lambda}(x)=z-1$ and the other coordinates remain unchanged;
- if $\sigma_{t^-}^{\xi,\lambda}(x-1)=\sigma_{t^-}^{\xi,\lambda}(x+1)=z-1=0$ and $U_{(x,z)}^{\downarrow}(n)\leq \frac{\lambda}{1+\lambda}$, let $\sigma_t^{\xi,\lambda}(x)=0$ and the other coordinates remain unchanged.

If neither of these two aforementioned conditions is satisfied, we do nothing.

Let $\mathbb{P}$ or $\mathbb{E}$ stand for the probability law corresponding to $\mathcal{T}^{\uparrow}$, $\mathcal{T}^{\downarrow}$, $\mathcal{U}^{\uparrow}$ and $\mathcal{U}^{\downarrow}$. Recall that μ_L^{λ} is the stationary probability measure for the dynamics. The dynamics $(\sigma_t^{\mu,\lambda})_{t\geq 0}$ is constructed by first taking the initial path ξ sampling from μ at $t=0$ and then using the graphical construction with parameter λ for $t>0$. This sampling is independent of $\mathbb{P}$. Define $P_t^{\mu,\lambda}(\cdot):=\mathbb{P}(\sigma_t^{\mu,\lambda}=\cdot)$, and likewise $P_t^{\mu,\lambda}(\mathcal{A}):=\mathbb{P}[\sigma_t^{\mu,\lambda}\in\mathcal{A}]$ for $\mathcal{A}\subset\Omega_L$. When it is clear in the context, we use the notations $(\sigma_t^{\mu})_{t\geq 0}$ and P_t^{μ}, ignoring the parameter λ.

This graphical construction allows us to construct all the trajectories $(\sigma_t^{\xi,\lambda})_{t\geq 0}$ starting from all $\xi\in\Omega_L$ and all parameters $\lambda\in[0,\infty)$ simultaneously. It preserves the order, affirmed in the following proposition. The proof of this proposition, which we omit, is almost identical to that of [10, Proposition 3.1].

Proposition 2.1. *Let ξ and ξ' be two elements of Ω_L satisfying $\xi\leq\xi'$, and $0\leq\lambda\leq\lambda'$. With the graphical construction above, we have*

$$\begin{aligned}&\mathbb{P}\big[\forall t\in[0,\infty):\sigma_t^{\xi,\lambda}\leq\sigma_t^{\xi',\lambda}\big]=1,\\&\mathbb{P}\big[\forall t\in[0,\infty):\sigma_t^{\xi,\lambda'}\leq\sigma_t^{\xi,\lambda}\big]=1.\end{aligned}\tag{2.2}$$

2.2. *Useful reclaimed results*

We have the asymptotic information about $Z_L(\lambda)$, which is:

Theorem 2.2 (Theorem 2.1 in [2]). *For every $\lambda\in[0,2)$, we have*

$$\lim_{L\to\infty}\frac{Z_L(\lambda)}{2^L L^{-3/2}}=C(\lambda),\tag{2.3}$$

where $C(\lambda)>0$ is a constant, only dependent on λ.

Furthermore, to understand the Glauber dynamics, it is important to understand how the generator $\mathcal{L}$ acts on the paths in Ω_L. Let us introduce the settings. For a function $g:[\![0,L]\!]\to\mathbb{R}$, the discrete Laplace operator Δ is defined as follows: for any $x\in[\![1,L-1]\!]$,

$$(\Delta g)_x:=\frac{1}{2}\big(g(x-1)+g(x+1)\big)-g(x).$$

Besides, we define a function $f:\Omega_L\mapsto\mathbb{R}$ to be $f(\xi):=\xi_x$, and let $\mathcal{L}\xi_x:=(\mathcal{L}f)(\xi)$ for $x\in[\![1,L-1]\!]$. Considering (1.6), we know that $\mathcal{L}\xi_x=\mu_L^{\lambda}(\xi_x|\xi_{x-1},\xi_{x+1})-\xi_x$, and a calculation yields the following identity which we recall as a lemma.

Lemma 2.3 (Lemma 2.3 in [2]). *For every $\lambda>0$ and every $x\in[\![1,L-1]\!]$, we have*

$$\mathcal{L}\xi_x=(\Delta\xi)_x+\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}}-\left(\frac{\lambda-1}{\lambda+1}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}.\tag{2.4}$$

3. Lower bound on the mixing time for $\lambda \in (0, 2)$

This section is devoted to providing a lower bound on the mixing time of the Glauber dynamics for $\lambda \in (0, 2)$, which is the following proposition.

Proposition 3.1. *For all* $\lambda \in (0, 2)$ *and all* $\epsilon \in (0, 1)$, *we have*

$$T_{\text{mix}}^{L,\lambda}(\epsilon) \geq \frac{1}{\pi^2} L^2 \log L - C(\lambda, \epsilon) L^2 =: t_{C(\lambda,\epsilon)}, \tag{3.1}$$

where $C(\lambda, \epsilon) > 0$ *is a constant, only dependent on* λ *and* ϵ.

Before we start the proof of Proposition 3.1, let us explain the idea. Note that the function $\Phi(\xi)$, defined in (3.2) below, is almost the area enclosed by the x-axis and the path $\xi \in \Omega_L$. Intuitively, $\Phi(\wedge)$ is of order L^2, while at equilibrium $\Phi(\xi)$ is of order $L^{3/2}$. The second moment method in [2, Theorem 3.2] does not supply a sharp lower bound on the mixing time. We adapt the idea in [2, Theorem 3.2] to provide the lower bound in (3.1) by proving the following.

(i) While the expected equilibrium value $\mu(\Phi)$ is at most of order $L^{3/2}$, $\mathbb{E}[\Phi(\sigma_t^{\wedge})]$ is much bigger than $L^{3/2}$ for all $t \leq t_{C(\lambda,\epsilon)}$;
(ii) On the one hand $\Phi(\sigma_t^{\mu})$ is fairly close to its mean $\mu(\Phi)$ by Markov's inequality, and on the other hand $\Phi(\sigma_t^{\wedge})$ is well concentrated around $\mathbb{E}[\Phi(\sigma_t^{\wedge})]$ by controlling its fluctuation through martingale approach.

Section 3.1 prepares all the ingredients for the first step of this strategy, and Section 3.2 is dedicated to the second step of the strategy, giving the proof of Proposition 3.1.

3.1. *Ingredients for the lower bound of the mixing time*

Inspired by [15, Equation (1)], Caputo *et al.* in [2, Equation (2.39)] defined the weighted area function $\Phi \colon \Omega_L \to \mathbb{R}$ by

$$\Phi(\xi) := \sum_{x=1}^{L-1} \xi_x \overline{\sin}(x), \tag{3.2}$$

where $\overline{\sin}(x) := \sin(\frac{\pi x}{L})$ and $\xi \in \Omega_L$. As [2, Equation (4.3)], we use Lemma 2.3 and summation by part to obtain

$$(\mathcal{L}\Phi)(\xi) = \sum_{x=1}^{L-1} \overline{\sin}(x) \mathcal{L}\xi_x = -\kappa_L \Phi(\xi) + \Psi(\xi), \tag{3.3}$$

where $\kappa_L := 1 - \cos(\frac{\pi}{L})$ and

$$\Psi(\xi) := \sum_{x=1}^{L-1} \overline{\sin}(x) \left[\mathbf{1}_{\{\xi_{x-1}=\xi_{x=1}=0\}} - \left(\frac{\lambda - 1}{\lambda + 1} \right) \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}} \right]. \tag{3.4}$$

Since $\overline{\sin}(x) \geq 0$ for all $x \in [\![0, L]\!]$, we have

$$\left|\Psi(\xi)\right| \leq \sum_{x=1}^{L-1} \overline{\sin}(x) \left[\mathbf{1}_{\{\xi_{x-1}=\xi_{x=1}=0\}} + \left| \frac{\lambda - 1}{\lambda + 1} \right| \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}} \right] =: \overline{\Psi}(\xi). \tag{3.5}$$

Caputo *et al.* gave an upper bound on $\mu_L^{\lambda}(\Phi)$. In [2, Equation (5.15)], they used coupling and monotonicity to obtain that for every positive integer k,

$$\sup_{\lambda \geq 0, L \in 2\mathbb{N}} \sup_{x \in [\![1, L-1]\!]} \mu_L^{\lambda} \left(\frac{(\xi_x)^k}{L^{k/2}} \right) < \infty.$$

Consequently, using $k = 1$ and $\overline{\sin}(x) \leq 1$, we have

$$\mu_L^{\lambda}(\Phi) \leq \sum_{x=1}^{L-1} \mu_L^{\lambda}(\xi_x) \leq c L^{3/2}, \tag{3.6}$$

where $c > 0$ does not depend on λ. In addition, Caputo *et al.* also gave a lower bound on $\mathbb{E}[\Phi(\sigma_t^\wedge)]$, which we recall as a lemma below.

Lemma 3.2 (Equation (5.24) in [2]). *For all $\lambda \in (0,2)$, all $t \geq 0$, all $L \geq 2$ and some constant $c(\lambda) > 0$, we have*

$$\mathbb{E}\left[\Phi\left(\sigma_t^\wedge\right)\right] \geq \Phi\left(\sigma_0^\wedge\right)e^{-\kappa_L t} - c(\lambda)L^{3/2}.$$

In view of (3.5), we need an upper bound on $\mathbb{P}[\sigma_t^\wedge(x-1) = \sigma_t^\wedge(x+1) \in \{0,1\}]$ for $x \in [\![1, L-1]\!]$, which is the following lemma.

Lemma 3.3. *For all $t \geq 0$, all $x \in [\![1, L-1]\!]$ and all $L \geq 2$, we have*

$$\mathbb{P}\left[\sigma_t^\wedge(x-1) = \sigma_t^\wedge(x+1) \in \{0,1\}\right] \leq C_1(\lambda)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}. \tag{3.7}$$

Proof. Since $\sigma_t^\wedge \geq \sigma_t^\mu$ for all $t \geq 0$, we know that for all $x \in [\![1, L-1]\!]$,

$$\begin{aligned}\mathbb{P}\left[\sigma_t^\wedge(x-1) = \sigma_t^\wedge(x+1) \in \{0,1\}\right] &\leq \mathbb{P}\left[\sigma_t^\mu(x-1) = \sigma_t^\mu(x+1) \in \{0,1\}\right] \\ &= \mu_L^\lambda\left(\xi_{x-1} = \xi_{x+1} \in \{0,1\}\right).\end{aligned}$$

For all $\lambda \in (0,2)$, all $x \in [\![1, L-1]\!] \cap 2\mathbb{N}$ and all $L \geq 2$, applying Theorem 2.2, we obtain

$$\mu_L^\lambda(\xi_x = 0) = \lambda\frac{Z_x(\lambda)Z_{L-x}(\lambda)}{Z_L(\lambda)} \leq C_2(\lambda)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}, \tag{3.8}$$

and

$$\mu_L^\lambda(\xi_{x-1} = \xi_{x+1} = 0) = \lambda^2\frac{Z_{x-1}(\lambda)Z_{L-x-1}(\lambda)}{Z_L(\lambda)} \leq C_2(\lambda)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}. \tag{3.9}$$

With the same conditions about λ, x and L above, as [2, Equation (5.23)] we have

$$\mu_L^\lambda(\xi_{x-1} = \xi_{x+1} = 1) = \frac{1+\lambda}{\lambda}\mu_L^\lambda(\xi_x = 0). \tag{3.10}$$

Therefore, by (3.8), (3.9) and (3.10), we obtain (3.7). □

3.2. *Proof of the lower bound on the mixing time*

Let us detail the second step of the aforementioned strategy. To prove that $\Phi(\sigma_t^\wedge)$ is well concentrated around its mean $\mathbb{E}[\Phi(\sigma_t^\wedge)]$, we do the following.

(i) For a fixed time t_0, we use the function $F(t,\xi) = \exp(\kappa_L(t-t_0))\Phi(\xi)$ to construct a Dynkin's martingale M (see [7, Lemma 5.1 in Appendix 1]).
(ii) To estimate the fluctuation of $F(t_0, \sigma_{t_0}^\wedge) = \Phi(\sigma_{t_0}^\wedge)$, we control the martingale bracket $\langle M.\rangle$ and the mean of $(\partial_t + \mathcal{L})F(t, \sigma_t^\wedge)$, which comes from the construction of Dynkin's martingale.

While $\Phi(\sigma_t^\mu)$ is at most of order $L^{3/2}$, $\Phi(\sigma_{t_0}^\wedge)$ is much bigger than $L^{3/2}$ for all $t_0 \leq t_{C(\lambda,\epsilon)}$. This property of Φ about σ_t^μ and $\sigma_{t_0}^\wedge$ can be used to provide a lower bound on the distance between μ and $P_{t_0}^\wedge$.

Proof of Proposition 3.1. We adapt the approach in [2, Proposition 5.3]. For $C \in (0,\infty)$, define

$$\mathcal{A}_C := \left\{\xi \in \Omega_L : \Phi(\xi) \leq CL^{3/2}\right\}. \tag{3.11}$$

Using Markov's inequality and (3.6), we obtain

$$1 - \mu(\mathcal{A}_C) = \mu\left(\Phi > CL^{3/2}\right) \leq \frac{\mu(\Phi)}{CL^{3/2}} \leq \frac{c}{C}, \tag{3.12}$$

where the rightmost term is smaller than or equal to $\epsilon/2$ for $C \geq 2c/\epsilon$. Our next step is to prove that for any given $\epsilon > 0$, if $t_0 \leq t_{C(\lambda,\epsilon)}$, we have

$$P_{t_0}^{\wedge}(\mathcal{A}_C) \leq \epsilon/2.$$

In order to obtain such an upper bound, we construct a Dynkin's martingale and control its fluctuation. Let t_0 be a fixed time, we define a function $F\colon [0, t_0] \times \Omega_L \to \mathbb{R}$ by

$$F(t, \xi) := e^{\kappa_L(t-t_0)}\Phi(\xi).$$

We recall that $\sigma_t^{\wedge}$, defined in Section 2.1, is the dynamics at time t starting with the maximal path $\wedge$. Further, we define a Dynkin's martingale by

$$M_t := F\big(t, \sigma_t^{\wedge}\big) - F\big(0, \sigma_0^{\wedge}\big) - \int_0^t (\partial_s + \mathcal{L})F\big(s, \sigma_s^{\wedge}\big)\,\mathrm{d}s. \tag{3.13}$$

Applying $(\mathcal{L}\Phi)(\xi) = -\kappa_L\Phi(\xi) + \Psi(\xi)$ in (3.3), we obtain

$$(\partial_t + \mathcal{L})F\big(t, \sigma_t^{\wedge}\big) = e^{\kappa_L(t-t_0)}\Psi\big(\sigma_t^{\wedge}\big). \tag{3.14}$$

For simplicity of notation, set

$$B(t) := \int_0^t e^{\kappa_L(s-t_0)}\Psi\big(\sigma_s^{\wedge}\big)\,\mathrm{d}s. \tag{3.15}$$

Now we give an upper bound on $\mathbb{E}[M_t^2]$ by controlling the martingale bracket $\langle M.\rangle$, which is such that the process $(M_t^2 - \langle M.\rangle_t)_{t\geq 0}$ is a martingale with respect to its natural filtration. Since there is at most one transition at each coordinate and each transition can change the value of M_t in absolute value by at most $2e^{\kappa_L(t-t_0)}$, we have

$$\partial_t\langle M.\rangle_t \leq \sum_{x=1}^{L-1} 4e^{2\kappa_L(t-t_0)} \leq 4Le^{2\kappa_L(t-t_0)}.$$

As $M_0 = 0$ and $\kappa_L = 1 - \cos(\frac{\pi}{L}) \geq \frac{\pi^2}{4L^2}$ for all $L \geq 4$, we obtain

$$\mathbb{E}\big[M_{t_0}^2\big] = \mathbb{E}\big[\langle M.\rangle_{t_0}\big] \leq \int_0^{t_0} 4Le^{2\kappa_L(t-t_0)}\,\mathrm{d}t \leq \frac{8L^3}{\pi^2}. \tag{3.16}$$

Furthermore, we give an upper bound for the mean of $B(t_0)$, defined in (3.15). Recalling the definitions of Ψ and $\overline{\Psi}$ in (3.4) and (3.5) respectively, we have

$$\begin{aligned}
\mathbb{E}\big[\big|B(t_0)\big|\big] &\leq \mathbb{E}\bigg[\int_0^{t_0} e^{\kappa_L(t-t_0)}\overline{\Psi}\big(\sigma_t^{\wedge}\big)\,\mathrm{d}t\bigg] \\
&\leq \mathbb{E}\bigg[\int_0^{t_0} e^{\kappa_L(t-t_0)}\overline{\Psi}\big(\sigma_t^{\mu}\big)\,\mathrm{d}t\bigg] \\
&\leq C_3(\lambda)\kappa_L^{-1}\sum_{x=1}^{L-1}\overline{\sin}(x)\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}} \\
&\leq C_4(\lambda)L^{3/2}.
\end{aligned} \tag{3.17}$$

The first inequality uses $|\Psi(\xi)| \leq \overline{\Psi}(\xi)$ for all $\xi \in \Omega_L$. The second inequality is due to two facts: (1) $\overline{\Psi}(\xi) \leq \overline{\Psi}(\xi')$ for $\xi \leq \xi'$; and (2) $\sigma_t^{\wedge} \geq \sigma_t^{\mu}$. In the third inequality, we use Fubini's Theorem to interchange the orders of integration and expectation, and use Lemma 3.3 to give an upper bound for $\mathbb{E}[\overline{\Psi}(\sigma_t^{\mu})]$. In the last inequality, we use the following inequality:

$$\overline{\sin}(x) = \sin\bigg(\frac{\pi x}{L}\bigg) \leq \frac{\min(x, L-x)\pi}{L}.$$

Here and now, we try to find the suitable small t_0 such that $\Phi(\sigma^{\wedge}_{t_0})$ is much larger than $L^{3/2}$ with high probability. We note that $\Phi(\sigma^{\wedge}_0) \ge \frac{1}{36}L^2$ and $\kappa_L \le \frac{\pi^2}{2L^2}$ for all $L \ge 2$. Let $C \ge 1$, and define

$$t_0 := \frac{1}{\pi^2}L^2 \log L - CL^2.$$

If $t_0 \le 0$, nothing needs to be done (for $L \le 4$, $t_0 \le 0$). In the remaining of this subsection, we assume $t_0 > 0$. Then for all $L \ge 2$, $t_0\kappa_L \le \frac{1}{2}\log L - C$. Moreover, there exists a universal constant $C_0 \ge 1$ such that if $C \ge C_0$, we have

$$\frac{1}{36}e^C \ge 3C.$$

By Lemma 3.2, for all $C \ge \max(C_0, c(\lambda))$, we have

$$\mathbb{E}\big[\Phi\big(\sigma^{\wedge}_{t_0}\big)\big] \ge 3CL^{3/2} - c(\lambda)L^{3/2} \ge 2CL^{3/2}.$$

Then, if $\Phi(\sigma^{\wedge}_{t_0}) \le CL^{3/2}$ (*i.e.* $\sigma^{\wedge}_{t_0} \in \mathcal{A}_C$, defined in (3.11)), it implies

$$\big|\Phi\big(\sigma^{\wedge}_{t_0}\big) - \mathbb{E}\big[\Phi\big(\sigma^{\wedge}_{t_0}\big)\big]\big| \ge CL^{3/2}$$

and

$$P^{\wedge}_{t_0}(\mathcal{A}_C) \le \mathbb{P}\big[\big|\Phi\big(\sigma^{\wedge}_{t_0}\big) - \mathbb{E}\big[\Phi\big(\sigma^{\wedge}_{t_0}\big)\big]\big| \ge CL^{3/2}\big]. \tag{3.18}$$

In addition, recalling $\Phi(\sigma^{\wedge}_{t_0}) = F(t_0, \sigma^{\wedge}_{t_0}) = M_{t_0} + F(0, \sigma^{\wedge}_0) + B(t_0)$ in (3.13) and using Markov's inequality, we obtain

$$\begin{aligned}
&\mathbb{P}\big[\big|\Phi\big(\sigma^{\wedge}_{t_0}\big) - \mathbb{E}\big[\Phi\big(\sigma^{\wedge}_{t_0}\big)\big]\big| \ge CL^{3/2}\big] \\
&\quad = \mathbb{P}\big[\big|M_{t_0} + B(t_0) - \mathbb{E}\big[B(t_0)\big]\big| \ge CL^{3/2}\big] \\
&\quad \le \mathbb{P}\left[|M_{t_0}| \ge \frac{1}{3}CL^{3/2}\right] + \mathbb{P}\left[\big|B(t_0)\big| \ge \frac{1}{3}CL^{3/2}\right] \\
&\quad \le \frac{9\mathbb{E}[M^2_{t_0}]}{C^2L^3} + \frac{3\mathbb{E}[|B(t_0)|]}{CL^{3/2}},
\end{aligned} \tag{3.19}$$

where the second last inequality holds for $C > 3C_4(\lambda)$ by $\mathbb{E}[|B(t_0)|] \le C_4(\lambda)L^{3/2}$ in (3.17). The last term in (3.19) is smaller than or equal to $\epsilon/2$ for $C \ge \max(\frac{18}{\pi\sqrt{\epsilon}}, \frac{12C_4(\lambda)}{\epsilon})$, on account of $\mathbb{E}[M^2_{t_0}] \le \frac{8L^3}{\pi^2}$ in (3.16) and (3.17). Combining (3.12), (3.18) and (3.19), we know that

$$\big\|P^{\wedge}_{t_0} - \mu\big\|_{\mathrm{TV}} \ge \mu(A_C) - P^{\wedge}_{t_0}(A_C) \ge 1 - \epsilon, \tag{3.20}$$

which holds for $C \ge \max\{\frac{2c}{\epsilon}, C_0, 3C_4(\lambda), \frac{18}{\pi\sqrt{\epsilon}}, \frac{12C_4(\lambda)}{\epsilon}\} =: C(\lambda, \epsilon)$. Therefore, for $C \ge C(\lambda, \epsilon)$, we have

$$T^{L,\lambda}_{\mathrm{mix}}(\epsilon) \ge \frac{1}{\pi^2}L^2 \log L - CL^2.$$

□

4. Upper bound on the mixing time for $\lambda \in (0, 1]$

This section is devoted to providing an upper bound on the mixing time of the dynamics for the regime $\lambda \in (0, 1]$. For any $\xi \in \Omega_L$, by the triangle inequality, we have

$$\big\|P^{\xi}_t - P^{\mu}_t\big\|_{\mathrm{TV}} \le \sum_{\xi' \in \Omega_L} \mu\big(\xi'\big)\big\|P^{\xi}_t - P^{\xi'}_t\big\|_{\mathrm{TV}} \le \max_{\xi' \in \Omega_L}\big\|P^{\xi}_t - P^{\xi'}_t\big\|_{\mathrm{TV}}. \tag{4.1}$$

To give an upper bound for the term in the rightmost hand side above, we use the following characterization of total variation distance. Let α and β be two probability measures on Ω_L. We say that ϑ is a coupling of α and β, if ϑ is a probability measure on $\Omega_L \times \Omega_L$ such that $\vartheta(\xi \times \Omega_L) = \alpha(\xi)$ and $\vartheta(\Omega_L \times \xi') = \beta(\xi')$ for any elements $\xi, \xi' \in \Omega_L$. The following proposition says that the total variation distance measures how well we can couple two random variables with distribution laws α and β respectively.

Proposition 4.1 (Proposition 4.7 [12]). *Let α and β be two probability distributions on Ω_L. Then*

$$\|\alpha-\beta\|_{\mathrm{TV}}=\inf\bigl\{\vartheta\bigl(\bigl\{(\xi,\xi'):\xi\neq\xi'\bigr\}\bigr):\vartheta \text{ is a coupling of } \alpha \text{ and } \beta\bigr\}.$$

The graphical construction in Section 2.1 provides a coupling between P_t^{ξ} and $P_t^{\xi'}$, which preserves the monotonicity asserted in Proposition 2.1. Therefore, σ_t^{ξ} lies between $\sigma_t^{\vee}$ and $\sigma_t^{\wedge}$ for any $\xi\in\Omega_L$. Applying Proposition 4.1, we obtain

$$\bigl\|P_t^{\xi}-P_t^{\xi'}\bigr\|_{\mathrm{TV}}\le\mathbb{P}\bigl[\sigma_t^{\xi}\neq\sigma_t^{\xi'}\bigr]\le\mathbb{P}\bigl[\sigma_t^{\wedge}\neq\sigma_t^{\vee}\bigr],\tag{4.2}$$

where the last inequality is due to the fact that after the dynamics starting from the two extremal paths have coalesced, we must have $\sigma_t^{\wedge}=\sigma_t^{\xi}=\sigma_t^{\vee}$ for any $\xi\in\Omega_L$. This argument was used in [2, Theorem 3.1] to obtain an upper bound on the mixing time. Comparing with the coupling in [2, Section 2.2.1], the graphical construction in Section 2.1 provides more independent flippable corners and maximizes the fluctuation of the area enclosed by $\sigma_t^{\wedge}$ and $\sigma_t^{\vee}$. Adapting the approach in [8, Section 7], we use a supermartingale approach to have a good control of the fluctuation of the area enclosed by $\sigma_t^{\wedge}$ and $\sigma_t^{\vee}$ to obtain a sharp upper bound on the mixing time. Let the coalescing time τ be

$$\tau:=\inf\bigl\{t\ge 0:\sigma_t^{\wedge}=\sigma_t^{\vee}\bigr\},$$

which is the first instant when the dynamics starting from the two extremal paths coalesce. By (4.1) and (4.2), we obtain

$$d^{L,\lambda}(t)\le\mathbb{P}\bigl[\sigma_t^{\wedge}\neq\sigma_t^{\vee}\bigr]=\mathbb{P}[\tau>t].\tag{4.3}$$

In this section, our goal is to show that for any given $\delta>0$ and all L sufficiently large, with high probability, we have

$$\tau\le\frac{1+\delta}{\pi^2}L^2\log L.$$

We adapt the approach in [8, Section 7] to achieve this goal. In practice, it is more feasible to couple two dynamics when, at least, one of them is at equilibrium. Let

$$\begin{aligned}\tau_1&:=\inf\bigl\{t\ge 0,\sigma_t^{\wedge}=\sigma_t^{\mu}\bigr\},\\ \tau_2&:=\inf\bigl\{t\ge 0,\sigma_t^{\vee}=\sigma_t^{\mu}\bigr\},\end{aligned}\tag{4.4}$$

where we recall that the dynamics $(\sigma_t^{\mu})_{t\ge 0}$ is constructed by first taking the initial path ξ by sampling μ at $t=0$ and then using the graphical construction for $t>0$. By the definition of τ, we know that

$$\tau=\max(\tau_1,\tau_2).$$

For this goal, it is sufficient to prove the following proposition.

Proposition 4.2. *For $i\in\{1,2\}$, any given $\lambda\in(0,1]$ and $\delta>0$, we have*

$$\lim_{L\to\infty}\mathbb{P}\biggl[\tau_i\le(1+\delta)\frac{1}{\pi^2}L^2\log L\biggr]=1.\tag{4.5}$$

Theorem 1.1 is proved as a combination of Proposition 3.1 and Proposition 4.2. Therefore, there is a cutoff in the Markov chains for $\lambda\in(0,1]$. Since the proofs about τ_1 and τ_2 in Proposition 4.2 are similar, we only give the proof of (4.5) for τ_1. For any given $\delta>0$, set

$$t_\delta:=(1+\delta)\frac{1}{\pi^2}L^2\log L.$$

We outline the idea for the proof. We define a weighted area function A_t in (4.9) below, which is almost the area enclosed by the paths $\sigma_t^{\wedge}$ and σ_t^{μ} at time t. Moreover, $(A_t)_{t\ge 0}$ is a supermartingale when $\lambda\in(0,1]$. Due to this, we obtain that at time $t_{\delta/2}=(1+\frac{\delta}{2})\frac{1}{\pi^2}L^2\log L$, $A_{t_{\delta/2}}$ is close to equilibrium. After time $t_{\delta/2}$, we estimate the fluctuation of $(A_t)_{t\ge t_{\delta/2}}$ by the supermartingale approach applying [8, Proposition 29], and then relate the time interval with the fluctuation to obtain (4.5).

4.1. *A weighted area function*

In this subsection, we define an area function A_t. First, inspired by [15, Equation (1)], we define a function $\overline{\Phi}_\beta \colon \Omega_L \to [0,\infty)$ given by

$$\overline{\Phi}_\beta(\xi) := \sum_{x=1}^{L-1} \xi_x \, \overline{\cos}_\beta(x),$$

where $\overline{\cos}_\beta(x) := \cos(\frac{\beta(x-L/2)}{L})$, and β is a constant in $(2\pi/3,\pi)$. The constant β is only dependent on δ and sufficiently close to π, which will be chosen in the proof of Lemma 4.4 below. We can see that $\overline{\Phi}_\beta(\xi)$ is approximately the area enclosed by the x-axis and the path $\xi \in \Omega_L$. Throughout this paper, we omit the index β in $\overline{\Phi}_\beta$ and $\overline{\cos}_\beta$ as much as possible. Observe that if ξ and ξ' are two elements of Ω_L satisfying $\xi \le \xi'$, then

$$\overline{\Phi}(\xi) \le \overline{\Phi}\bigl(\xi'\bigr). \tag{4.6}$$

The minimal increment of the function $\overline{\Phi}$ is

$$\delta_{\min} := \min_{\xi \le \xi', \xi \ne \xi'} \bigl(\overline{\Phi}\bigl(\xi'\bigr) - \overline{\Phi}(\xi)\bigr) = 2\cos\biggl(\frac{\beta(L/2-1)}{L}\biggr), \tag{4.7}$$

and

$$2\cos\biggl(\frac{\beta(L/2-1)}{L}\biggr) \ge \frac{1}{2}(\pi-\beta) \tag{4.8}$$

for $L \ge 6$ and $\beta \in (2\pi/3,\pi)$, where we use the inequality $\cos(\pi/2-x) = \sin x \ge x/2$ for $x \in [0,\pi/3]$. Let the weighted area function $A \colon [0,\infty) \mapsto [0,\infty)$ be

$$A_t := \frac{\overline{\Phi}(\sigma_t^\wedge) - \overline{\Phi}(\sigma_t^\mu)}{\delta_{\min}}. \tag{4.9}$$

We observe that τ_1, defined in (4.4), is the first time at which A_t reaches zero. Moreover, A_t equals to zero if and only if $\sigma_t^\wedge$ equals to σ_t^μ. If $\tau_1 \le t_{\delta/2}$, we are done. In the rest of this section, we assume $\tau_1 > t_{\delta/2}$.

Take $\eta > 0$ and sufficiently small, and $K := \lceil 1/(2\eta) \rceil$. We define a sequence of successive stopping times $(\mathcal{T}_i)_{i=2}^K$ by

$$\mathcal{T}_2 := \inf\bigl\{t \ge t_{\delta/2} : A_t \le L^{\frac{3}{2}-2\eta}\bigr\},$$

and for $3 \le i \le K$,

$$\mathcal{T}_i := \inf\bigl\{t \ge \mathcal{T}_{i-1} : A_t \le L^{\frac{3}{2}-i\eta}\bigr\}.$$

For consistency of notations, we set $\mathcal{T}_\infty := \max(\tau_1, t_{\delta/2})$. The remaining of this section is devoted to proving the following proposition.

Proposition 4.3. *Given $\delta > 0$, if η is chosen to be a sufficiently small positive constant with $K = \lceil 1/(2\eta) \rceil > 1/(2\eta)$, we have*

$$\lim_{L\to\infty} \mathbb{P}\Biggl[\{\mathcal{T}_2 = t_{\delta/2}\} \cap \Biggl(\bigcap_{i=3}^K \bigl\{\Delta\mathcal{T}_i \le 2^{-i}L^2\bigr\}\Biggr) \cap \bigl\{\mathcal{T}_\infty - \mathcal{T}_K \le L^2\bigr\}\Biggr] = 1,$$

where $\Delta\mathcal{T}_i := \mathcal{T}_i - \mathcal{T}_{i-1}$ for $3 \le i \le K$.

If Proposition 4.3 holds, for L sufficiently large, we have

$$\tau_1 = \mathcal{T}_\infty \le t_{\delta/2} + \sum_{i=3}^K 2^{-i}L^2 + L^2 \le (1+\delta)\frac{1}{\pi^2}L^2 \log L.$$

Then Proposition 4.2 is proved. The idea for Proposition 4.3 is from [8, Section 7] as follows:

1. We first show that the decay rate of A_t is at least $1-\cos(\frac{\pi}{L})$, and then we obtain $\mathcal{T}_2 = t_{\delta/2}$ with high probability.
2. During the time interval $[\mathcal{T}_{i-1}, \mathcal{T}_i]$ for $3 \le i \le K$, we apply the supermartingale approach ([8, Proposition 29]) to show that with high probability
$$\langle A_.\rangle_{\mathcal{T}_i} - \langle A_.\rangle_{\mathcal{T}_{i-1}} \le L^{3-2(i-1)\eta+\frac{1}{2}\eta}.$$
Similarly for the time interval $[\mathcal{T}_K, \mathcal{T}_\infty]$, we apply [8, Proposition 29] to show that with high probability
$$\langle A_.\rangle_{\mathcal{T}_\infty} - \langle A_.\rangle_{\mathcal{T}_K} \le L^2.$$
3. We compare $\mathcal{T}_\infty - \mathcal{T}_K$ with $\langle A_.\rangle_{\mathcal{T}_\infty} - \langle A_.\rangle_{\mathcal{T}_K}$. As $\partial_t\langle A_.\rangle \ge 1$ for all $t < \mathcal{T}_\infty$, we have
$$\mathcal{T}_\infty - \mathcal{T}_K \le \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} \partial_t\langle A_.\rangle\,\mathrm{d}t = \langle A_.\rangle_{\mathcal{T}_\infty} - \langle A_.\rangle_{\mathcal{T}_K}.$$
For $3 \le i \le K$, to compare $\langle A_.\rangle_{\mathcal{T}_i} - \langle A_.\rangle_{\mathcal{T}_{i-1}}$ with $\mathcal{T}_i - \mathcal{T}_{i-1}$, we provide a better lower bound on $\partial_t\langle A_.\rangle$ in terms of the highest point of $\sigma_t^\wedge$ and the maximal length of a monotone segment of σ_t^μ in Lemma 4.7.
4. We use induction method to show that $\mathcal{T}_i - \mathcal{T}_{i-1} \le 2^{-i}L^2$ for all $i \in [\![3, K]\!]$, arguing by contradiction.

4.2. *The proof of* $\mathcal{T}_2 = t_{\delta/2}$

The main task of this subsection is to prove that the function A_t has a contraction property, due to which we obtain $\mathcal{T}_2 = t_{\delta/2}$ with high probability. Above all, we want to understand how the generator $\mathcal{L}$ acts on the function $\overline{\Phi}$. We have

$$(\mathcal{L}\overline{\Phi})(\xi) = \sum_{x=1}^{L-1} \overline{\cos}(x)\mathcal{L}\xi_x.$$

We recall Lemma 2.3: for any $\xi \in \Omega_L$,

$$\mathcal{L}\xi_x = (\Delta\xi)_x + \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} + \left(\frac{1-\lambda}{1+\lambda}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}}.$$

For $\xi, \xi' \in \Omega_L$, we have

$$\sum_{x=1}^{L-1} \overline{\cos}(x)\big((\Delta\xi')_x - (\Delta\xi)_x\big) = -\left(1-\cos\left(\frac{\beta}{L}\right)\right)\sum_{x=1}^{L-1}\overline{\cos}(x)\big(\xi'_x - \xi_x\big). \tag{4.10}$$

Considering

$$\mathcal{L}\xi_x - (\Delta\xi)_x = \mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=0\}} + \left(\frac{1-\lambda}{1+\lambda}\right)\mathbf{1}_{\{\xi_{x-1}=\xi_{x+1}=1\}},$$

we see that both terms in the right-hand side are nonnegative and monotonically decreasing in ξ for $\lambda \in (0,1]$. Hence, if $\xi \le \xi'$, we know that

$$\mathcal{L}\xi_x - (\Delta\xi)_x \ge \mathcal{L}\xi'_x - \big(\Delta\xi'\big)_x. \tag{4.11}$$

For simplicity of notation, we set

$$\gamma = \gamma_{L,\beta} := 1 - \cos(\beta/L).$$

On the grounds of Lemma 2.3, (4.10) and (4.11), if $\xi \le \xi'$, we obtain

$$\begin{aligned}
(\mathcal{L}\overline{\Phi})\big(\xi'\big) - (\mathcal{L}\overline{\Phi})(\xi) &= \sum_{x=1}^{L-1}\overline{\cos}(x)\Big(\big(\Delta\xi'\big)_x - (\Delta\xi)_x + \big(\mathcal{L}\xi'_x - \big(\Delta\xi'\big)_x\big) - \big(\mathcal{L}\xi_x - (\Delta\xi)_x\big)\Big) \\
&\le \sum_{x=1}^{L-1}\overline{\cos}(x)\big((\Delta\xi)'_x - (\Delta\xi)_x\big) \\
&\le -\gamma\big(\overline{\Phi}\big(\xi'\big) - \overline{\Phi}(\xi)\big).
\end{aligned} \tag{4.12}$$

Now we are ready to prove that $\mathcal{T}_2 = t_{\delta/2}$ with high probability.

Lemma 4.4. *For all $\epsilon > 0$, all sufficiently small $\delta > 0$ and $0 < \eta < \delta/10$, if L is sufficiently large, we have*

$$\mathbb{P}[\mathcal{T}_2 > t_{\delta/2}] \le \epsilon.$$

Proof. By $\sigma_t^\wedge \ge \sigma_t^\mu$ and (4.12), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[\overline{\Phi}\big(\sigma_t^\wedge\big) - \overline{\Phi}\big(\sigma_t^\mu\big)\big] &= \mathbb{E}\big[(\mathcal{L}\overline{\Phi})\big(\sigma_t^\wedge\big) - (\mathcal{L}\overline{\Phi})\big(\sigma_t^\mu\big)\big] \\ &\le -\gamma\mathbb{E}\big[\overline{\Phi}\big(\sigma_t^\wedge\big) - \overline{\Phi}\big(\sigma_t^\mu\big)\big].\end{aligned} \tag{4.13}$$

Using (4.13), $\overline{\Phi}(\sigma_0^\wedge) \le \frac{1}{2}L^2$ and $\overline{\Phi}(\xi) \ge 0$ for all $\xi \in \Omega_L$, we obtain

$$\mathbb{E}\big[\overline{\Phi}\big(\sigma_t^\wedge\big) - \overline{\Phi}\big(\sigma_t^\mu\big)\big] \le e^{-\gamma t}\big(\overline{\Phi}\big(\sigma_0^\wedge\big) - \overline{\Phi}\big(\sigma_0^\mu\big)\big) \le \frac{1}{2}L^2 e^{-\gamma t}. \tag{4.14}$$

Thus, applying Markov's inequality, we achieve

$$\begin{aligned}\mathbb{P}[\mathcal{T}_2 > t_{\delta/2}] &= \mathbb{P}\big[A_{t_{\delta/2}} > L^{\frac{3}{2}-2\eta}\big] \\ &\le \frac{1}{2\delta_{\min}} L^{2\eta+\frac{1}{2}} e^{-\gamma t_{\delta/2}},\end{aligned} \tag{4.15}$$

where the last inequality uses (4.14) and the definition of A_t in (4.9). Recalling $\gamma = 1-\cos(\beta/L)$ and using the inequality $1 - \cos x \ge \frac{1}{2}x^2 - \frac{1}{24}x^4$ for all $x \ge 0$, we have

$$\gamma t_{\delta/2} \ge \frac{\beta^2}{2\pi^2}\left(1 + \frac{\delta}{2}\right)\log L - \frac{\beta^4}{24L^2}\left(1 + \frac{\delta}{2}\right)\log L.$$

For $\delta > 0$ sufficiently small and $0 < \eta < \delta/10$, we choose

$$\beta = \pi\sqrt{\frac{1 + \frac{9}{20}\delta}{1 + \frac{\delta}{2}}} \in (2\pi/3, \pi)$$

which satisfies

$$\frac{1}{2}\left(1 + \frac{\delta}{2}\right)\frac{\beta^2}{\pi^2} = \frac{1}{2} + \frac{9}{40}\delta > \frac{1}{2} + 2\eta.$$

With this choice of β, the rightmost term of (4.15) vanishes as L tends to infinity. □

4.3. *The estimation of $\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}$*

Due to Dynkin's martingale formula, we know that

$$A_t - A_0 - \int_0^t \mathcal{L}A_s \,\mathrm{d}s$$

is a martingale. Moreover, we let $\langle A.\rangle_t$ represent the predictable bracket associated with this martingale. The objective of this subsection is to show that $\langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}$ is small for all $i \in [\![3, K]\!]$. For any $i \in [\![3, K]\!]$, let

$$\Delta_i\langle A\rangle := \langle A.\rangle_{\mathcal{T}_i} - \langle A.\rangle_{\mathcal{T}_{i-1}}, \tag{4.16}$$

and let

$$\Delta_\infty\langle A\rangle := \langle A.\rangle_{\mathcal{T}_\infty} - \langle A.\rangle_{\mathcal{T}_K}. \tag{4.17}$$

We have $\mathcal{L}A_s \le 0$, according to (4.12), $\sigma_t^\wedge \ge \sigma_t^\mu$, and the monotonicity of the function $\overline{\Phi}$ stated in (4.6). Then, A_t is a supermartingale for $\lambda \in (0, 1]$. Its jump amplitudes in absolute value are bounded below by 1 for $t < \tau_1$ where the absorption time τ_1 is defined in (4.4). Moreover, for $t < \tau_1$ we can always find one flippable corner in $\sigma_t^\wedge$ and one in σ_t^μ

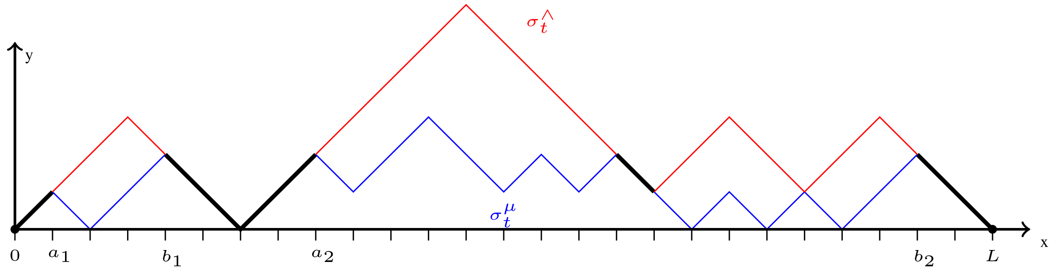

Fig. 2. In this figure, $\sigma_t^\wedge$ consists of the red line segments and black thick line segments, while σ_t^μ consists of the blue line segments and black thick line segments. Moreover, $\mathcal{B}_t = [\![a_1, b_1]\!] \cup [\![a_2, b_2]\!]$, $\#(\mathcal{D}_t \cap [\![a_1, b_1]\!]) = 3$, and $\#(\mathcal{D}_t \cap [\![a_2, b_2]\!]) = 13$. In $[\![a_2, b_2]\!]$, the monotone segments of σ_t^μ are $[\![a_2, a_2+1]\!]$, $[\![a_2+1, a_2+3]\!]$, $[\![a_2+3, a_2+5]\!]$, and so on as shown in the figure.

which can change the value of A_t, and the total rates of these two corners are at least 1. Therefore, the jump rates of A_t are least 1 for $t < \tau_1$. We refer to Figure 2 for illustration: those flippable corners in σ_t^μ and $\sigma_t^\wedge$ which are not totally colored black can change the value of A_t, and the total rates of these corners are at least 1. Now, we are in the setting to apply [8, Proposition 29] which, under some condition, allows to control hitting times of supermartingales in terms of the martingale bracket.

Proposition 4.5 (Proposition 29 in [8]). *Let $(\mathbf{M}_t)_{t\ge 0}$ be a pure-jump supermartingale with bounded jump rates and jump amplitudes, and $\mathbf{M}_0 \le a$ almost surely. Let $\langle \mathbf{M}.\rangle$, with an abuse of notation, denote the predictable bracket associated with the martingale $\overline{\mathbf{M}}_t = \mathbf{M}_t - I_t$ where I is the compensator of $\mathbf{M}$. Given $b \in \mathbb{R}$ and $b \le a$, we set*

$$\tau_b := \inf\{t \ge 0 : \mathbf{M}_t \le b\}.$$

If the amplitudes of the jumps of $(\mathbf{M}_t)_{t\ge 0}$ are bounded above by $a-b$, for any $u \ge 0$, we have

$$\mathbb{P}\big[\langle \mathbf{M}.\rangle_{\tau_b} \ge (a-b)^2 u\big] \le 8u^{-1/2}. \tag{4.18}$$

Now we apply Proposition 4.5 to prove that the event

$$\mathcal{A}_L := \big\{\forall i \in [\![3, K]\!], \Delta_i\langle A\rangle \le L^{3-2(i-1)\eta+\frac{1}{2}\eta}\big\} \cap \big\{\Delta_\infty\langle A\rangle \le L^2\big\}$$

has almost the full mass, which is the following lemma.

Lemma 4.6. *We have*

$$\lim_{L\to\infty} \mathbb{P}[\mathcal{A}_L] = 1. \tag{4.19}$$

Proof. We just need to show that the probability of its complement $\mathcal{A}_L^{\complement}$ is almost zero. We apply Proposition 4.5 to $(A_{t+\mathcal{T}_{i-1}})_{t\ge 0}$ with $a_i = L^{\frac{3}{2}-(i-1)\eta}$ and $b_i = L^{\frac{3}{2}-i\eta}$. For every $i \in [\![3, K]\!]$, we obtain

$$\mathbb{P}\big[\Delta_i\langle A\rangle \ge \big(L^{\frac{3}{2}-(i-1)\eta} - L^{\frac{3}{2}-i\eta}\big)^2 u_i\big] \le 8u_i^{-\frac{1}{2}}, \tag{4.20}$$

where we choose $u_i = L^{\frac{1}{2}\eta}(1-L^{-\eta})^{-2}$, satisfying

$$\big(L^{\frac{3}{2}-(i-1)\eta} - L^{\frac{3}{2}-i\eta}\big)^2 u_i = L^{3-2(i-1)\eta+\frac{1}{2}\eta}.$$

We see that u_i tends to infinity as L tends to infinity. Accordingly, the rightmost term in (4.20) vanishes as L tends to infinity.

We apply Proposition 4.5 to $(A_{t+\mathcal{T}_K})_{t\ge 0}$ with $a_\infty = L^{\frac{3}{2}-K\eta}$ and $b_\infty = 0$. We choose u_∞ such that $(a_\infty - b_\infty)^2 u_\infty = L^2$, *i.e.*

$$u_\infty = L^{-1+2K\eta},$$

which tends to infinity due to $K = \lceil 1/(2\eta) \rceil > 1/(2\eta)$. Thus $\mathbb{P}[\Delta_\infty \langle A \rangle \geq L^2]$ tends to zero as L tends to infinity. Since K is a constant, we have

$$\lim_{L\to\infty} \mathbb{P}\big[\mathcal{A}_L^{\complement}\big] = 0. \qquad \square$$

4.4. *The comparison of $\mathcal{T}_i - \mathcal{T}_{i-1}$ to $\Delta_i \langle A \rangle$*

As explained in Section 4.3, we have $\partial_t \langle A. \rangle \geq 1$ for all $t < \mathcal{T}_\infty$. Therefore, we obtain

$$\Delta_\infty \langle A \rangle = \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} \partial_t \langle A. \rangle \, \mathrm{d}t \geq \int_{\mathcal{T}_K}^{\mathcal{T}_\infty} 1 \, \mathrm{d}t = \mathcal{T}_\infty - \mathcal{T}_K.$$

Hence, when the event $\mathcal{A}_L$ holds, we obtain

$$\mathcal{T}_\infty - \mathcal{T}_K \leq \Delta_\infty \langle A \rangle \leq L^2.$$

Now we control the intermediate increment $\mathcal{T}_i - \mathcal{T}_{i-1}$ for $3 \leq i \leq K$. To do that, we compare $\mathcal{T}_i - \mathcal{T}_{i-1}$ with $\langle A. \rangle_{\mathcal{T}_i} - \langle A. \rangle_{\mathcal{T}_{i-1}} = \Delta_i \langle A. \rangle$. First, we give a lower bound on $\partial_t \langle A. \rangle$, which is related with: (a) the maximal contribution among all the coordinates $x \in [\![0, L]\!]$ in the definition of A_t; and (b) the amount of flippable corners in σ_t^μ or $\sigma_t^\wedge$ that can change the value of A_t. Considering the definition of A_t in (4.9), set

$$H(t) := \max_{x \in [\![0, L]\!]} \sigma_t^\wedge(x). \tag{4.21}$$

For a lower bound on the quantity mentioned in (b), we need the maximal length of the monotone segment of σ_t^μ. For $\xi \in \Omega_L$, we define

$$Q_1(\xi) := \max\big\{n \geq 1, \exists i \in [\![0, L-n]\!], \forall x \in [\![i+1, i+n]\!], \xi_x - \xi_{x-1} = 1\big\},$$
$$Q_2(\xi) := \max\big\{n \geq 1, \exists i \in [\![0, L-n]\!], \forall x \in [\![i+1, i+n]\!], \xi_x - \xi_{x-1} = -1\big\},$$

and

$$Q(\xi) := \max\big(Q_1(\xi), Q_2(\xi)\big). \tag{4.22}$$

Using these two quantities $H(t)$ and $Q(\sigma_t^\mu)$, we obtain a lower bound for $\partial_t \langle A. \rangle$, which is the following lemma.

Lemma 4.7. *We have*

$$\partial_t \langle A. \rangle \geq \max\left(1, \frac{\lambda \delta_{\min} A_t}{3(1+\lambda) H(t) Q(\sigma_t^\mu)}\right). \tag{4.23}$$

Proof. We observe that A_t displays a jump whenever either σ_t^μ or $\sigma_t^\wedge$ flips a corner. Note that by (4.9) and (4.7), any jump amplitude in absolute value of A is at least 1. Since any flippable corner is flipped with rate at least

$$\min\left\{\frac{1}{2}, \frac{1}{1+\lambda}, \frac{\lambda}{1+\lambda}\right\} = \frac{\lambda}{1+\lambda},$$

we obtain

$$\partial_t \langle A. \rangle_t \geq \frac{\lambda}{1+\lambda} \#\big\{x \in \mathcal{B}_t : \Delta \sigma_t^\mu(x) \neq 0\big\},$$

where $\mathcal{B}_t := \{x \in [\![1, L-1]\!] : \exists y \in [\![x-1, x+1]\!], \sigma_t^\wedge(y) \neq \sigma_t^\mu(y)\}$. For simplicity of notation, set $\mathcal{D}_t := \{x \in \mathcal{B}_t : \Delta \sigma_t^\mu(x) \neq 0\}$. Let $[\![a, b]\!]$ denote the horizontal coordinates of a maximal connected component of $\mathcal{B}_t$, for which we refer to Figure 2 for illustration. Since σ_t^μ can not be monotone in the entire domain $[\![a, b]\!]$, we know that

$$\#\big(\mathcal{D}_t \cap [\![a, b]\!]\big) \geq 1.$$

In $\mathcal{B}_t$, we decompose the path associated with σ_t^μ into consecutive maximal monotone segments. Then we know that in $\mathcal{B}_t$ every two consecutive components correspond to one flippable corner, which is a point in $\mathcal{D}_t$. As any maximal monotone component is at most of length $Q(\sigma_t^\mu)$ defined in (4.22), we obtain

$$\#\big(\mathcal{D}_t \cap [\![a,b]\!]\big) \geq \frac{1}{2}\left\lfloor \frac{b-a}{Q(\sigma_t^\mu)} \right\rfloor \geq \frac{1}{3}\frac{b-a}{Q(\sigma_t^\mu)}. \tag{4.24}$$

In addition, we observe that

$$\sum_{x=a}^{b} \frac{(\sigma_t^\wedge(x) - \sigma_t^\mu(x))\,\overline{\cos}(x)}{\delta_{\min}} \leq (b-a)\frac{H(t)}{\delta_{\min}}, \tag{4.25}$$

where $H(t)$ is defined in (4.21). Summing up all such intervals $[\![a,b]\!]$ and using (4.24) and (4.25), we obtain

$$A_t \leq \frac{3}{\delta_{\min}} H(t) Q\big(\sigma_t^\mu\big)\#\mathcal{D}_t.$$

Therefore, we have

$$\partial_t \langle A.\rangle \geq \frac{\lambda}{1+\lambda}\#\mathcal{D}_t \geq \frac{\lambda\delta_{\min}}{3(1+\lambda)}\frac{A_t}{H(t)Q(\sigma_t^\mu)}.$$

This yields the desired result. □

To give a good lower bound for $\partial_t\langle A.\rangle$, we need to control $Q(\sigma_t^\mu)$ and $H(t)$. Our next step is to give an upper bound on $Q(\sigma_t^\mu)$, which is the following lemma. We recall the notation

$$t_\delta = (1+\delta)\frac{1}{\pi^2}L^2\log L.$$

Lemma 4.8. *We have*

$$\lim_{L\to\infty} \mathbb{P}\big[\exists t \in [0, t_\delta] : Q\big(\sigma_t^\mu\big) > (\log L)^2\big] = 0. \tag{4.26}$$

Proof. Firstly, we prove that there exists a constant $C(\lambda) > 0$ such that for all $L \geq 2$

$$\mu\big(Q(\xi) > (\log L)^2\big) \leq 2C(\lambda)L^{5/2}2^{-(\log L)^2}. \tag{4.27}$$

Since there are at most L starting positions for a monotone segment either monotonically increasing or decreasing, we have

$$\#\big\{\xi \in \Omega_L : Q(\xi) > (\log L)^2\big\} \leq L2^{1+L-(\log L)^2}.$$

Moreover, as $\lambda^{\mathcal{N}(\xi)} \leq 1$ for $\lambda \in (0,1]$ and any $\xi \in \Omega_L$, we obtain

$$\mu\big(Q(\xi) > (\log L)^2\big) \leq C_5(\lambda)\frac{L2^{1+L-(\log L)^2}}{2^L L^{-3/2}} = 2C_5(\lambda)L^{5/2}2^{-(\log L)^2}, \tag{4.28}$$

where we use the inequality $Z_L(\lambda) \geq C_5(\lambda)^{-1}2^L L^{-3/2}$ for all $L \geq 2$ and some $C_5(\lambda) > 0$ by Theorem 2.2. Secondly, since there are at most L corners in any path $\xi \in \Omega_L$, we have

$$\sum_{x=1}^{L-1} R_x(\xi) \leq L,$$

where $R_x(\xi)$ is defined in (1.5). Therefore, for any subset $\mathcal{A} \subset \Omega_L$ and $s \geq 0$,

$$\mathbb{P}\big[\forall t \in \big[s, s+L^{-1}\big] : \sigma_t^\mu \in \mathcal{A} \mid \sigma_s^\mu \in \mathcal{A}\big] \geq e^{-1}. \tag{4.29}$$

Taking $\mathcal{A} := \{\xi \in \Omega_L : Q(\xi) > (\log L)^2\}$, we define the occupation time to be

$$u(t) := \int_0^t \mathbf{1}_{\mathcal{A}}\big(\sigma_s^\mu\big)\,\mathrm{d}s. \tag{4.30}$$

By Fubini's Theorem, we obtain

$$\mathbb{E}\big[u(2t_\delta)\big] = 2t_\delta \mu(\mathcal{A}). \tag{4.31}$$

Using (4.29) and strong Markov property, we give a lower bound for $\mathbb{E}[u(2t_\delta)]$:

$$\mathbb{E}\big[u(2t_\delta)\big] \geq e^{-1}L^{-1}\mathbb{P}\big[\exists t \in [0, t_\delta] : \sigma_t^\mu \in \mathcal{A}\big]. \tag{4.32}$$

By (4.31), (4.32) and (4.27), we have

$$\mathbb{P}\big[\exists t \in [0, t_\delta] : \sigma_t^\mu \in \mathcal{A}\big] \leq 2eLt_\delta\mu(\mathcal{A}) \leq 4eC_5(\lambda)L^{7/2}t_\delta 2^{-(\log L)^2}, \tag{4.33}$$

which vanishes as L tends to infinity. Therefore, we conclude the proof. □

The last ingredient for the proof of Proposition 4.3 is to control $H(t)$, defined in (4.21). Recall that $t_\delta = (1+\delta)\frac{1}{\pi^2}L^2\log L$.

Lemma 4.9. *We have*

$$\lim_{L\to\infty}\sup_{t\in[t_{\delta/2},t_\delta]}\mathbb{P}\big[H(t) \geq 2L^{\frac{1}{2}}(\log L)^2\big] = 0. \tag{4.34}$$

Intuitively, for $\lambda \in (0,2)$, $(\frac{\xi_{[xL]}}{\sqrt{L}})_{x\in[0,1]}$ under μ_L^λ converges to Brownian excursion. (A rough argument for the intuition goes as follows. By Equation (3.8), we have

$$\mu_L^\lambda\big(\exists x \in [\![L^{1/3}, L-L^{1/3}]\!] : \xi_x = 0\big) \leq 2c(\lambda)\sum_{x=L^{1/3}}^{L/2}\frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}} \leq c'(\lambda)L^{-1/6}.$$

For $\xi \in \Omega_L$, define

$$\mathbf{L}(\xi) := \sup\{x \leq L/2 : \xi_x = 0\},$$
$$\mathbf{R}(\xi) := \inf\{x \geq L/2 : \xi_x = 0\},$$

and we observe that

$$\mu_L^\lambda(\cdot|\mathbf{L}=\ell,\mathbf{R}=r) = \mu_\ell^\lambda \otimes \mathbf{P}\Big(\cdot\big|\min_{1\leq i<r-\ell}S_i > 0;\, S_{r-\ell}=0\Big) \otimes \mu_{L-r}^\lambda,$$

where $\mathbf{P}$ denotes the law of the symmetric nearest-neighbor simple random walk on $\mathbb{Z}$. As $(\frac{S_{[x(r-\ell)]}}{\sqrt{r-\ell}})_{x\in[0,1]}$ under the law $\mathbf{P}(\cdot|\min_{1\leq i<r-\ell}S_i > 0;\, S_{r-\ell}=0)$ converges to the Brownian excursion, we conclude the proof.) Therefore, the dynamics $(\sigma_t^\wedge)_{t\geq 0}$ is like the simple exclusion process, and we can apply [10, Theorem 2.4] to obtain Lemma 4.9. We postpone the proof in Appendix A. Now, we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. We define the event $\mathcal{H}_L$ where the highest point of $\sigma_t^\wedge$ is not too high and there are a lot of flippable corners in σ_t^μ during the time interval $[t_{\delta/2}, t_{\delta/2}+L^2]$,

$$\mathcal{H}_L = \bigg\{\int_{t_{\delta/2}}^{t_{\delta/2}+L^2}\mathbf{1}_{\{H(t)\leq 2L^{\frac{1}{2}}(\log L)^2\}\cap\{Q(\sigma_t^\mu)\leq(\log L)^2\}}\,\mathrm{d}t \geq L^2\big(1-2^{-(K+1)}\big)\bigg\}.$$

First, we show that $\mathcal{H}_L$ holds with high probability. We have

$$\begin{aligned}\mathbb{P}\big[\mathcal{H}_L^{\complement}\big] &= \mathbb{P}\bigg[\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{H(t)>2L^{\frac{1}{2}}(\log L)^2\}\cup\{Q(\sigma_t^\mu)>(\log L)^2\}}\,\mathrm{d}t \geq L^2 2^{-(K+1)}\bigg] \\ &\leq \mathbb{P}\bigg[\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{H(t)>2L^{\frac{1}{2}}(\log L)^2\}}\,\mathrm{d}t \geq L^2 2^{-(K+2)}\bigg] \\ &\quad + \mathbb{P}\bigg[\int_{t_{\delta/2}}^{t_{\delta/2}+L^2} \mathbf{1}_{\{Q(\sigma_t^\mu)>(\log L)^2\}}\,\mathrm{d}t \geq L^2 2^{-(K+2)}\bigg],\end{aligned} \tag{4.35}$$

which vanishes as L tends to infinity, grounded on Markov's inequality, Lemma 4.8, Lemma 4.9 and the fact that K is a constant.

From now on, we assume the event $\mathcal{A}_L \cap \mathcal{H}_L \cap \{\mathcal{T}_2 = t_{\delta/2}\}$. Based on (4.35), Lemma 4.6 and Lemma 4.8, we have

$$\lim_{L\to\infty} \mathbb{P}\big[\mathcal{A}_L \cap \mathcal{H}_L \cap \{\mathcal{T}_2 = t_{\delta/2}\}\big] = 1.$$

By induction, we show that $\Delta\mathcal{T}_j = \mathcal{T}_j - \mathcal{T}_{j-1} \leq 2^{-j}L^2$ for all $j \in [\![3, K]\!]$. We argue by contradiction: let i_0 be the smallest integer satisfying

$$\Delta\mathcal{T}_{i_0} > 2^{-i_0}L^2.$$

We know that

$$\Delta_{i_0}\langle A\rangle \geq \int_{\mathcal{T}_{i_0-1}}^{\mathcal{T}_{i_0-1}+2^{-i_0}L^2} \partial_t\langle A.\rangle \mathbf{1}_{\{H(t)\leq 2L^{\frac{1}{2}}(\log L)^2\}\cap\{Q(\sigma_t^\mu)\leq(\log L)^2\}}\,\mathrm{d}t. \tag{4.36}$$

According to Lemmas 4.7, 4.8 and 4.9, we have a lower bound for $\partial_t\langle A.\rangle$ when the indicator function equals to 1. That bound is

$$\partial_t\langle A.\rangle \geq \frac{\lambda\delta_{\min}}{3(1+\lambda)}\frac{A_t}{H(t)Q(\sigma_t^\mu)} \geq \frac{\lambda\delta_{\min}}{6(1+\lambda)}\frac{A_t}{L^{\frac{1}{2}}(\log L)^4}. \tag{4.37}$$

Since $\mathcal{T}_2 = t_{\delta/2}$ and $\Delta\mathcal{T}_j = \mathcal{T}_j - \mathcal{T}_{j-1} \leq 2^{-j}L^2$ for $j < i_0$, we know that

$$\mathcal{T}_{i_0-1} \leq t_{\delta/2} + L^2\sum_{j=3}^{i_0-1} 2^{-j} \leq t_{\delta/2} + \big(1 - 2^{-(i_0-1)}\big)L^2,$$

and then $\mathcal{T}_{i_0-1} + 2^{-i_0}L^2 \leq t_{\delta/2} + L^2$. Moreover, when the assumption $\mathcal{H}_L$ holds, the indicator function

$$\mathbf{1}_{\{H(t)\leq 2L^{\frac{1}{2}}(\log L)^2\}\cap\{Q(\sigma_t^\mu)\leq(\log L)^2\}}$$

is equal to 1 on a set, which is of Lebesgue measure at least

$$\big(2^{-i_0} - 2^{-(K+1)}\big)L^2 \geq 2^{-(K+1)}L^2. \tag{4.38}$$

Combining (4.36), (4.37) and (4.38), we obtain

$$\Delta_{i_0}\langle A\rangle \geq 2^{-(K+1)}L^2\frac{\lambda\delta_{\min}}{6(1+\lambda)}\frac{A_t}{L^{\frac{1}{2}}(\log L)^4} \geq 2^{-(K+1)}\frac{\lambda\delta_{\min}}{6(1+\lambda)}L^{3-i_0\eta}(\log L)^{-4}, \tag{4.39}$$

where the last inequality uses the fact that $A_t > L^{\frac{3}{2}-i_0\eta}$, for $t < \mathcal{T}_{i_0}$. In addition, since we are in $\mathcal{A}_L$, we know that

$$\Delta_{i_0}\langle A\rangle \leq L^{3-2(i_0-1)\eta+\frac{1}{2}\eta}. \tag{4.40}$$

However, as $i_0 \geq 3$, we have

$$3 - 2(i_0-1)\eta + \frac{1}{2}\eta < 3 - i_0\eta.$$

Therefore, there is a contradiction between (4.39) and (4.40), as long as L is large enough. □

5. Upper bound on the mixing time of the dynamics starting from the extremal paths for $\lambda \in (1, 2)$

For the pinning model without positive constraint (see [2, Section 1]), the critical value is $\lambda_c = 1$, while the critical value is $\lambda_c = 2$ for the pinning model with positive constraint. Due to the repulsion effect of the impenetrable wall, the process $(A_t)_{t\geq 0}$ defined in Section 4.3 is not a supermartingale for $\lambda \in (1, 2)$. But there is still monotonicity in the dynamics starting with the maximal (or minimal) path for $\lambda \in (1, 2)$, which can be exploited to provide an upper bound on the mixing time by applying the censoring inequality in [14, Theorem 1.1]. This inequality says that canceling some prescribed updates slows down the mixing of the Glauber dynamics starting from the maximal (or minimal) configuration of a monotone spin system.

Let us state the setting for applying the censoring inequality. A censoring scheme is a càdlàg function defined by

$$\mathcal{C} : \mathbb{R}^+ \to \mathcal{P}(\Theta),$$

where Θ is defined in (2.1) and $\mathcal{P}(\Theta)$ is the set of all subsets of Θ. The censored dynamics with a censoring scheme $\mathcal{C}$ is the dynamics obtained from the graphical construction in Section 2.1, except that the update at time t is canceled if and only if it is an element of $\mathcal{C}(t)$. In other words, we construct the dynamics by using the graphical construction in Section 2.1 with one extra rule: if $\mathcal{T}^{\uparrow}_{(x,z)}$ or $\mathcal{T}^{\downarrow}_{(x,z)}$ rings at time t, the update is performed if and only if $(x, z) \notin \mathcal{C}(t)$. Let $(\sigma_t^{\xi,\mathcal{C}})_{t\geq 0}$ denote the trajectory of the censored dynamics with a censoring scheme $\mathcal{C}$ and starting from the path $\xi \in \Omega_L$, and let $P_t^{\xi,\mathcal{C}}$ denote the law of distribution of the time marginal $\sigma_t^{\xi,\mathcal{C}}$.

The Glauber dynamics of this polymer pinning model is a monotone spin system in the sense of [14, Section 1.1] (detailed in Appendix B), and we refer to Figure 5 in Appendix B for a quick look. The following proposition follows directly from [14, Theorem 1.1].

Proposition 5.1. *For any prescribed censoring scheme $\mathcal{C}$, for all $\lambda \in [0, \infty)$, all $t \geq 0$ and $\xi \in \{\wedge, \vee\}$, we have*

$$\|P_t^{\xi} - \mu\|_{\mathrm{TV}} \leq \|P_t^{\xi,\mathcal{C}} - \mu\|_{\mathrm{TV}}. \tag{5.1}$$

Besides Proposition 5.1, we need the two following results in the proof of the upper bound on the mixing time. Firstly, by [12, Lemmas 20.5 and 20.11], we know that the asymptotic rate of convergence to equilibrium of this reversible Markov chain is

$$\lim_{t\to\infty} t^{-1} \log d^{L,\lambda}(t) = -\mathrm{gap}_{L,\lambda}, \tag{5.2}$$

where $\mathrm{gap}_{L,\lambda} > 0$ is the spectral gap defined in (1.12). By monotonicity of the Glauber dynamics and (4.3), for all $\lambda > 0$ we have

$$d^{L,\lambda}(t) \leq \mathbb{P}\big(\sigma_t^{\wedge} \neq \sigma_t^{\vee}\big) = \mathbb{P}\left(\Phi\big(\sigma_t^{\wedge}\big) - \Phi\big(\sigma_t^{\vee}\big) \geq 2\sin\left(\frac{\pi}{L}\right)\right), \tag{5.3}$$

where $\Phi(\xi)$ is defined in (3.2). Moreover, for all $\lambda > 0$, by [2, Equation (4.1)] we have

$$\mathbb{E}\big[\Phi\big(\sigma_t^{\wedge}\big)\big] - \mathbb{E}\big[\Phi\big(\sigma_t^{\vee}\big)\big] \leq \frac{L^2}{2} e^{-t\kappa_L}.$$

Applying Markov's inequality, we reclaim the useful result in [2].

Lemma 5.2. *For all $\lambda > 0$, we have*

$$d^{L,\lambda}(t) \leq \frac{L^2 e^{-\kappa_L t}}{4\sin(\frac{\pi}{L})}. \tag{5.4}$$

Plugging this into (5.2), we obtain

$$\mathrm{gap}_{L,\lambda} \geq \kappa_L = 1 - \cos\left(\frac{\pi}{L}\right). \tag{5.5}$$

Secondly, the following lemma is an application of the Cauchy–Schwarz inequality and the reversibility of the Markov chain. For reference, we mention [1, Equation (2.6)].

Lemma 5.3. *For any probability distribution ν on Ω_L, we have*

$$\|\nu P_t - \mu\|_{\mathrm{TV}} \le \frac{1}{2} e^{-t\cdot \mathrm{gap}_{L,\lambda}} \sqrt{\mathrm{Var}_\mu(\rho)}, \tag{5.6}$$

where $\rho := \frac{d\nu}{d\mu}$ *and* $\mathrm{Var}_\mu(\rho) := \mu(\rho^2) - \mu(\rho)^2$.

We define

$$G_L := \{(x,1) : x \in [\![2, L-2]\!] \cap 2\mathbb{N}\}, \tag{5.7}$$

where $\mathcal{T}^{\uparrow}_{(x,1)}$ or $\mathcal{T}^{\downarrow}_{(x,1)}$ rings, the update – in the graphical construction of Section 2.1 – changes the number of contact points $\mathcal{N}$, defined in (1.1). Moreover, G_L corresponds to the centers of the green squares shown in Figure 5. Before we start the proof of the upper bound on the mixing time for the dynamics starting with the maximal path $\wedge$, we outline the idea for Proposition 5.4 with $\lambda \in (1,2)$.

(i) We elaborate a censoring scheme $\mathcal{C}$, where $\mathcal{C}(t) = G_L$ for $t < t_{\delta/2}$ and $\mathcal{C}(t) = \varnothing$ for $t \ge t_{\delta/2}$. Therefore the dynamics $(\sigma_t^{\wedge,\mathcal{C}})_{0\le t<t_{\delta/2}}$ does not touch the x-axis except at the two coordinates $x = 0, L$.
(ii) By Remark 1 and Theorem 1.1, the distribution of $\sigma^{\wedge,\mathcal{C}}_{t_{\delta/2}}$ is close to μ_L^0 in total variation distance.
(iii) As the Radon–Nikodym derivative of μ_L^0 with respect to μ_L^λ is bounded by a constant, we apply Lemma 5.3 and use (5.5) to conclude the proof.

Proposition 5.4. *For any $\lambda \in (1,2)$, any $\epsilon > 0$ and any $\delta > 0$, if L is sufficiently large, we have*

$$T_{\mathrm{mix}}^{L,\wedge}(\epsilon) \le \frac{1+\delta}{\pi^2} L^2 \log L. \tag{5.8}$$

Proof. Recall that $\mathcal{N}$ is the number of contact points, defined in (1.1). We run the dynamics starting from the maximal path $\wedge$, censoring those updates which change the value of contact points $\mathcal{N}$ for $t < t_{\delta/2}$. More precisely, recalling $t_\delta = (1+\delta)\frac{1}{\pi^2} L^2 \log L$, we present a censoring scheme $\mathcal{C} : \mathbb{R}^+ \to \mathcal{P}(\Theta)$, defined by

$$\mathcal{C}(t) := \begin{cases} G_L & \text{if } t \in [0, t_{\delta/2}), \\ \varnothing & \text{if } t \in [t_{\delta/2}, \infty). \end{cases}$$

We recall that $\sigma_t^{\wedge,\mathcal{C}}$ is the dynamics constructed by using the graphical construction with one extra rule: when the clock process $\mathcal{T}^{\uparrow}_{(x,1)}$ or $\mathcal{T}^{\downarrow}_{(x,1)}$ rings for any $x \in [\![1, L-1]\!] \cap 2\mathbb{N}$ and all $t < t_{\delta/2}$, we do not update. We refer to Figure 3 for illustration. While $t \ge t_{\delta/2}$, $(\sigma_t^{\wedge,\mathcal{C}})_{t\ge t_{\delta/2}}$ is constructed by the graphical construction in Section 2.1 without censoring.

Now we show that $P^{\wedge,\mathcal{C}}_{t_{\delta/2}}$ is close to μ_L^0. By Remark 1, applying Theorem 1.1, for all $\lambda \in (1,2)$, all $\delta > 0$ and all $\epsilon > 0$, if L is sufficiently large, we have

$$\left\| P^{\wedge,\mathcal{C}}_{t_{\delta/2}} - \mu_L^0 \right\|_{\mathrm{TV}} \le \epsilon/2. \tag{5.9}$$

For any $\xi \in \Omega_L$, define

$$\rho(\xi) := \frac{d\mu_L^0}{d\mu_L^\lambda}(\xi),$$

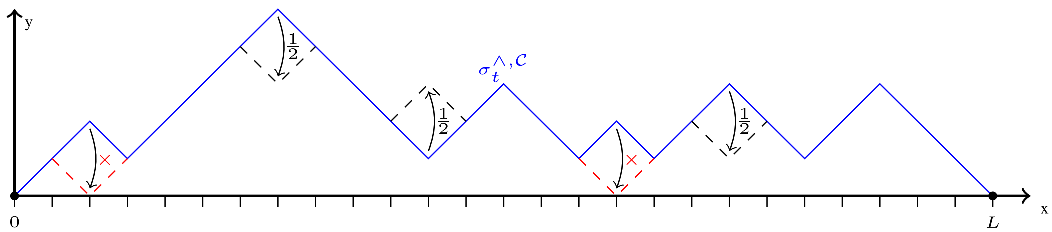

Fig. 3. A graphical representation of the jump rates for the dynamics $\sigma_t^{\wedge,\mathcal{C}}$ when $t < t_{\delta/2}$. Those red dashed corners are not available and labeled with $\times$, while the other corners are flippable with rate $1/2$.

and we want to show that ρ is bounded above uniformly for $\xi \in \Omega_L$. For any $\xi \in \Omega_L \setminus \Omega_L^+$ – recalling $\Omega_L^+ = \{\xi \in \Omega_L : \mathcal{N}(\xi) = 0\}$, since $\mu_L^0(\xi) = 0$,

$$\rho(\xi) = \frac{\mu_L^0(\xi)}{\mu_L^\lambda(\xi)} = 0.$$

While for any $\xi \in \Omega_L^+$, applying Theorem 2.2, for all $L \geq 4$ we have

$$\rho(\xi) = \frac{d\mu_L^0}{d\mu_L^\lambda}(\xi) = \frac{\mu_L^0(\xi)}{\mu_L^\lambda(\xi)} = \frac{1/Z_{L-2}(1)}{1/Z_L(\lambda)} \leq C_5(\lambda),$$

where $C_5(\lambda) > 0$ is a suitable constant and only depends on λ. By Lemma 5.3 and (5.5), for any given $\delta > 0$, we have

$$\lim_{L\to\infty} \left\| \mu_L^0 P_{\frac{\delta}{2}L^2 \log L} - \mu_L^\lambda \right\|_{\mathrm{TV}} = 0. \tag{5.10}$$

At this moment, we are ready to show that $P_{t_\delta}^{\wedge,\mathcal{C}}$ – the distribution of the censored dynamics at t_δ – is close to the stationary measure μ_L^λ. By the definition of $\mathcal{C}$, we have

$$\begin{aligned} \left\| P_{t_\delta}^{\wedge,\mathcal{C}} - \mu_L^\lambda \right\|_{\mathrm{TV}} &= \left\| P_{t_{\delta/2}}^{\wedge,\mathcal{C}} P_{t_\delta - t_{\delta/2}} - \mu_L^\lambda \right\|_{\mathrm{TV}} \\ &\leq \left\| P_{t_{\delta/2}}^{\wedge,\mathcal{C}} P_{t_\delta - t_{\delta/2}} - \mu_L^0 P_{t_\delta - t_{\delta/2}} \right\|_{\mathrm{TV}} + \left\| \mu_L^0 P_{t_\delta - t_{\delta/2}} - \mu_L^\lambda \right\|_{\mathrm{TV}} \\ &\leq \left\| P_{t_{\delta/2}}^{\wedge,\mathcal{C}} - \mu_L^0 \right\|_{\mathrm{TV}} + \left\| \mu_L^0 P_{t_\delta - t_{\delta/2}} - \mu_L^\lambda \right\|_{\mathrm{TV}}. \end{aligned} \tag{5.11}$$

Here the first inequality uses the triangle inequality. The second inequality is based on the fact that $\|\alpha P_t - \beta P_t\|_{\mathrm{TV}} \leq \|\alpha - \beta\|_{\mathrm{TV}}$ for any two probability measures α, β on Ω_L, and P_t is a transition matrix on Ω_L. The first term in (5.11) is not bigger than $\epsilon/2$ by (5.9) for L sufficiently large. The second term in (5.11) is smaller than or equal to $\epsilon/2$ by (5.10) for L sufficiently large.

Recall that $P_t^\wedge$ is the distribution of $\sigma_t^\wedge$ without censoring. By Proposition 5.1, for any $t \geq 0$, we have

$$\left\| P_t^\wedge - \mu_L^\lambda \right\|_{\mathrm{TV}} \leq \left\| P_t^{\wedge,\mathcal{C}} - \mu_L^\lambda \right\|_{\mathrm{TV}}. \tag{5.12}$$

Combining (5.11) and (5.12), we conclude the proof. □

Our next task is to provide an upper bound on the mixing time for the dynamics starting from the minimal path.

Proposition 5.5. *For any $\lambda \in (1,2)$, any $\epsilon > 0$ and any $\delta > 0$, if L is sufficiently large, we have*

$$T_{\mathrm{mix}}^{L,\vee}(\epsilon) \leq \frac{1+\delta}{\pi^2} L^2 \log L. \tag{5.13}$$

The idea for the proof of Proposition 5.5 for $\lambda \in (1,2)$ is similar to Proposition 5.4:

(i) We first show that under $P_{s_0(L)}^\vee$ with $s_0(L) := 10L^{16/9} \log L$ which is the marginal distribution of $\sigma_{s_0(L)}^\vee$, with high probability $\sigma_{s_0(L)}^\vee$ does not touch the x-axis in the interval $[\![M, L-M]\!]$ for some M sufficiently large.

(ii) For the time interval $[s_0(L), s_0(L) + t_{\delta/2})$, let $(\sigma_t^{\vee,\mathcal{C}})_{s_0(L) \leq t < s_0(L)+t_{\delta/2}}$ denote the dynamics censoring those updates which can change the number of contact points.

(iii) By Remark 1 and Theorem 1.1, roughly speaking, the distribution of $\sigma_{s_0(L)+t_{\delta/2}}^{\vee,\mathcal{C}}$ is close to μ_L^0 in total variation distance. Then we repeat the (iii) step stated above Proposition 5.4 to conclude the proof.

Lemma 5.6. *For any given $\epsilon > 0$ and $\lambda \in (1,2)$, let $M = M(\lambda, \epsilon)$ be a positive integer, and*

$$\mathcal{E}_{L,M} := \left\{ \xi \in \Omega_L : \xi_x \geq 1, \forall x \in [\![M, L-M]\!] \right\}. \tag{5.14}$$

For all $L \geq 2M$, we have

$$\mathbb{P}\left[\sigma_{s_0}^\vee \in \mathcal{E}_{L,M} \right] \geq 1 - \epsilon/2. \tag{5.15}$$

Proof. Let m and n be two positive integers, and $n < m < L/2$. Observe that in the graphical construction, if we run the dynamics $(\sigma_t^{\vee})_{t\geq 0}$ with the points $(2n, 0)$ and $(2m, 0)$ fixed for all $t \geq 0$, denoted as $(\overline{\sigma}_t^{\vee})_{t\geq 0}$ with $\overline{\sigma}_t^{\vee}(2n) \equiv \overline{\sigma}_t^{\vee}(2m) \equiv 0$ for all $t \geq 0$, we have

$$\forall t \geq 0, \quad \overline{\sigma}_t^{\vee} \leq \sigma_t^{\vee}. \tag{5.16}$$

By symmetry, to give an upper bound on $\mathbb{P}[\sigma_t^{\vee}(x) = 0]$, we only need to consider $x \in [\![0, L/2]\!]$. For all $M \leq x \leq L/2$ and $x \in 2\mathbb{N}$, let $\bar{x} := 2\lfloor x^{8/9}/2 \rfloor$ and $\bar{L} := 2\lfloor L^{8/9}/2 \rfloor$. For all L sufficiently large, by (5.4) we obtain

$$T_{\mathrm{mix}}^{L,\lambda}\big(L^{-3/2}\big) \leq \frac{18}{\pi^2} L^2 \log L. \tag{5.17}$$

Therefore, the quantity s_0 satisfies

$$T_{\mathrm{mix}}^{\bar{L},\lambda}\big(\bar{L}^{-3/2}\big) \leq s_0.$$

Using (5.16), (5.17) and (3.8) respectively, for all $t \geq s_0$, we take $2n := x - \overline{x}$ and $2m := x + \overline{x}$ in (5.16) to obtain

$$\begin{aligned}\mathbb{P}\big[\sigma_t^{\vee}(x) = 0\big] &\leq \mathbb{P}\big[\overline{\sigma}_t^{\vee}(x) = 0\big] \\ &\leq \mu_{2\bar{x}}^{\lambda}(\xi_{\bar{x}} = 0) + \big\| P_t^{\vee} - \mu_{2\bar{x}}^{\lambda} \big\|_{\mathrm{TV}} \leq C_6(\lambda) x^{-4/3},\end{aligned} \tag{5.18}$$

where $C_6(\lambda) > 0$ only depends on λ. In the second inequality, there is an abuse of notation – $P_t^{\vee}$ denotes the distribution of $\sigma_t^{\vee}$ starting with the minimal path $\vee$ of $\Omega_{2\bar{x}}$. Therefore, due to symmetry and (5.18), we obtain

$$\begin{aligned}\sum_{x=M}^{L-M} \mathbb{P}\big[\sigma_{s_0}^{\vee}(x) = 0\big] &= 2 \sum_{x=M}^{L/2} \mathbb{P}\big[\sigma_{s_0}^{\vee}(x) = 0\big] \\ &\leq 2C_7(\lambda) M^{-1/3}.\end{aligned} \tag{5.19}$$

Let $C(\lambda, \epsilon) > 0$ be a constant such that the right-hand side is smaller than $\epsilon/2$, if $M \geq C(\lambda, \epsilon)$. Applying Markov's inequality and (5.19), we obtain

$$\mathbb{P}\big[\sigma_{s_0}^{\vee} \notin \mathcal{E}_{L,M}\big] = \mathbb{P}\Bigg[\sum_{x=M}^{L-M} \mathbf{1}_{\{\sigma_{s_0}^{\vee}(x)=0\}} \geq 1\Bigg] \leq \epsilon/2. \tag{5.20}$$

□

For the dynamics starting from $\xi \in \mathcal{E}_{L,M}$, we censor the updates that change the number of the contact points until time $t_{\delta/2}$. Then we show that its distribution at time $t_{3\delta/4}$ is close to μ_L^{λ} in total variation distance.

Lemma 5.7. *Let $\xi \in \mathcal{E}_{L,M}$, and let $(\sigma_t^{\xi,\mathcal{C}})_{t\geq 0}$ be a censored dynamics with the censoring scheme $\mathcal{C}\colon \mathbb{R}^+ \to \mathcal{P}(\Theta)$ defined by*

$$\mathcal{C}(t) := \begin{cases} G_L & \text{if } t \in [0, t_{\delta/2}), \\ \varnothing & \text{if } t \in [t_{\delta/2}, \infty). \end{cases}$$

where G_L is defined in (5.7). For any given $\epsilon > 0$, for all L sufficiently large, we have

$$\big\| P_{t_{3\delta/4}}^{\xi,\mathcal{C}} - \mu_L^{\lambda} \big\|_{\mathrm{TV}} < \epsilon/2, \tag{5.21}$$

where we recall that $t_\delta = (1 + \delta)\pi^{-2} L^2 \log L$ and $P_t^{\xi,\mathcal{C}}$ denotes the marginal distribution of the censored dynamics $(\sigma_t^{\xi,\mathcal{C}})_{t\geq 0}$ at time t.

With Lemma 5.7 at hand, we are ready to prove Proposition 5.5. Combining Lemma 5.6, Lemma 5.7 and Proposition 5.1, we conclude the proof of Proposition 5.5, since $s_0 + t_{3\delta/4} \leq t_\delta$.

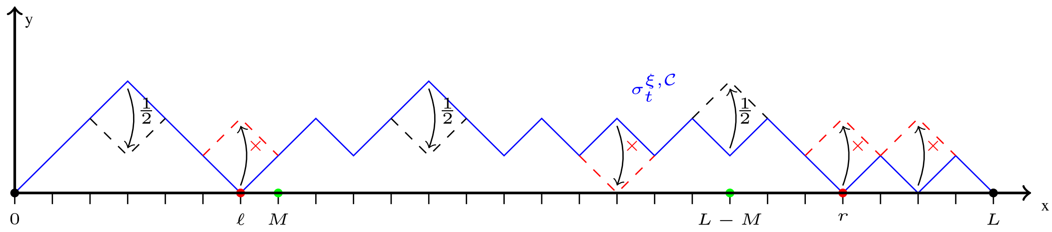

Fig. 4. A graphical representation of the jump rates for the censored dynamics $(\sigma_t^{\xi,\mathcal{C}})_{0\le t<t_{\delta/2}}$ starting from $\xi \in \mathcal{E}_{L,M}$. The red dashed corners are not available corners, labeled with $\times$. To the left hand side of the green point $(M,0)$, the red point $(\ell,0)$ is the first contact point with the x-axis at time $t=0$. Moreover, the corner at $(\ell,0)$ is fixed for $t\in[0,t_{\delta/2})$. Likewise, the same phenomenon holds for the green point $(L-M,0)$ and the red point $(r,0)$. In the time interval $[0,t_{\delta/2})$, the censored dynamics $(\sigma_t^{\xi,\mathcal{C}})_{0\le t<t_{\delta/2}}$ does not touch the x-axis in the interval $[\![\ell+1,r-1]\!]$.

Proof of Lemma 5.7. For $\xi\in\mathcal{E}_{L,M}$, set

$$\begin{aligned}\ell(\xi)&:=\sup\{x\le M:\xi_x=0\},\\ r(\xi)&:=\inf\{x\ge L-M:\xi_x=0\}.\end{aligned}\tag{5.22}$$

Observe that the censored dynamics $(\sigma_t^{\xi,\mathcal{C}})_{0\le t<t_{\delta/2}}$ restricted in the intervals $[\![0,\ell]\!]$, $[\![\ell,r]\!]$ and $[\![r,L]\!]$ respectively are independent. Let the marginal distribution restricted in these three intervals be denoted by $P_{t,\ell}^{\xi,\mathcal{C}}$, $P_{t,r-\ell}^{\xi,\mathcal{C}}$, $P_{t,L-r}^{\xi,\mathcal{C}}$ respectively. We refer to Figure 4 for illustration.

Let the censored dynamics restricted in the interval $[\![\ell,r]\!]$ be denoted by $(\widetilde{\sigma}_t^{\xi})_{t<t_{\delta/2}}$, whose invariant probability measure is $\mu_{r-\ell}^0$ defined in (1.2). By Theorem 1.1 and Remark 1, for given $\delta>0$ and $\epsilon>0$, for all L sufficiently large, we have

$$\|P_{t_{\delta/2},r-\ell}^{\xi,\mathcal{C}}-\mu_{r-\ell}^0\|_{\mathrm{TV}}\le\epsilon/4.\tag{5.23}$$

Note that the upper bound in (5.23) does not depend on the value of (ℓ,r). Moreover, observe that for any $\xi'\in\Omega_L$, the product distribution $P_{t_{\delta/2},l}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}$ satisfies

$$\big(P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}\big)(\xi')\le\frac{1}{Z_{r-\ell}(0)},\tag{5.24}$$

while $\mu_L^\lambda(\xi')\ge 1/Z_L(\lambda)$ since $\lambda\in(1,2)$. Therefore, for all $L>2M$ and for any $\xi'\in\Omega_L$, we have

$$\frac{\mathrm{d}P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}}{\mathrm{d}\mu_L^\lambda}(\xi')\le C_8(\lambda)2^{2M},\tag{5.25}$$

where the last inequality uses Theorem 2.2 and $r-\ell\ge L-2M$, since $\xi\in\mathcal{E}_{L,M}$. Note that the right-most hand side in (5.25) does not depend on the value of (ℓ,r), and the distribution of $\sigma_{t_{\delta/2}}^{\xi,\mathcal{C}}$ is

$$P_{t_{\delta/2}}^{\xi,\mathcal{C}}=P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes P_{t_{\delta/2},r-\ell}^{\xi,\mathcal{C}}\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}},\tag{5.26}$$

instead of $P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}$. Due to (5.23), we repeat the same procedure in (5.11) to obtain

$$\begin{aligned}\|P_{t_{3\delta/4}}^{\xi,\mathcal{C}}-\mu_L^\lambda\|&=\|P_{t_{\delta/2}}^{\xi,\mathcal{C}}P_{t_{3\delta/4}-t_{\delta/2}}-\mu_L^\lambda\|\\&\le\|P_{t_{\delta/2}}^{\xi,\mathcal{C}}P_{t_{3\delta/4}-t_{\delta/2}}-\big(P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}\big)P_{t_{3\delta/4}-t_{\delta/2}}\|\\&\quad+\|\big(P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}\big)P_{t_{3\delta/4}-t_{\delta/2}}-\mu_L^\lambda\|.\end{aligned}\tag{5.27}$$

Moreover, we have

$$\begin{aligned}&\|P_{t_{\delta/2}}^{\xi,\mathcal{C}}P_{t_{3\delta/4}-t_{\delta/2}}-\big(P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}\big)P_{t_{3\delta/4}-t_{\delta/2}}\|\\&\quad\le\|P_{t_{\delta/2}}^{\xi,\mathcal{C}}-P_{t_{\delta/2},\ell}^{\xi,\mathcal{C}}\otimes\mu_{r-\ell}^0\otimes P_{t_{\delta/2},L-r}^{\xi,\mathcal{C}}\|=\|P_{t_{\delta/2},r-\ell}^{\xi,\mathcal{C}}-\mu_{r-\ell}^0\|\le\varepsilon/4,\end{aligned}\tag{5.28}$$

where we have used (5.26) in the equality and (5.23) in the last inequality. While for the last term in (5.27), by (5.25) and Lemma 5.3, for all L sufficiently large we have

$$\left\| \left(P^{\xi,\mathcal{C}}_{t_{\delta/2},\ell} \otimes \mu^0_{r-\ell} \otimes P^{\xi,\mathcal{C}}_{t_{\delta/2},L-r}\right) P_{t_{3\delta/4}-t_{\delta/2}} - \mu^\lambda_L \right\| \le \varepsilon/4. \tag{5.29}$$

Combining (5.28) with (5.29), we conclude the proof. □

Theorem 1.2 is a combination of Proposition 3.1, Proposition 5.4, and Proposition 5.5.

Appendix A: Proof of Lemma 4.9

We lift the maximal path $\wedge$ up by a height $L^{1/2}(\log L)^2$. To be precise, define $\overline{\wedge} := \wedge + m$, *i.e.* $\overline{\wedge}_x = \wedge_x + m$ for all $x \in [\![0, L]\!]$, where $m := 2\lceil L^{1/2}(\log L)^2/2\rceil$. The graphical construction in Section 2.1, with Θ changed to be

$$\Theta' := \left\{(x,z) : x \in [\![1, L-1]\!], z \in [\![1, m + L/2 - 1 - |x - L/2|]\!], x + z \in 2\mathbb{N} + 1\right\},$$

allows us to couple the three dynamics $(\sigma^{\wedge,\lambda}_t)_{t\ge 0}$, $(\sigma^{\overline{\wedge},\lambda}_t)_{t\ge 0}$ and $(\sigma^{\overline{\wedge},0}_t)_{t\ge 0}$, starting from $\wedge$, $\overline{\wedge}$ and $\overline{\wedge}$ respectively, with parameter λ, λ and 0 respectively. By the monotonicity of the starting paths and the parameters λ in the dynamics, asserted in Proposition 2.1, we have

$$\begin{aligned} &\sigma^{\wedge,\lambda}_t \le \sigma^{\overline{\wedge},\lambda}_t, \\ &\sigma^{\overline{\wedge},\lambda}_t \le \sigma^{\overline{\wedge},0}_t. \end{aligned}$$

Set

$$\overline{H}(t) := \max_{x\in[\![0,L]\!]} \sigma^{\overline{\wedge},0}_t(x).$$

Since $\overline{H}(t) \ge H(t)$, it is enough to prove that

$$\lim_{L\to\infty} \mathbb{P}\left[\exists t \in [t_{\delta/2}, t_\delta] : \overline{H}(t) \ge 2L^{1/2}(\log L)^2\right] = 0, \tag{A.1}$$

where we recall that $t_\delta = (1+\delta)\frac{1}{\pi^2}L^2\log L$. We obtain such an upper bound in (A.1) by comparing $(\sigma^{\overline{\wedge},0}_t)_{t\ge 0}$ with the symmetric simple exclusion process.

A.1. *Simple exclusion process*

Define

$$\mathcal{S}_L := \left\{\zeta \in \mathbb{Z}^{L+1} : \zeta_0 = \zeta_L = m; |\xi_{x+1} - \xi_x| = 1, \forall x \in [\![0, L-1]\!]\right\}, \tag{A.2}$$

and

$$\mathcal{S}^+_L := \left\{\zeta \in \mathcal{S}_L : \zeta_x \ge 1, \forall x \in [\![0, L]\!]\right\}.$$

We define a Markov chain on $\mathcal{S}_L$ by specifying its generator $\mathfrak{L}$. The generator $\mathfrak{L}$ is defined by its action on the functions $\mathbb{R}^{\mathcal{S}_L}$,

$$(\mathfrak{L}f)(\zeta) := \frac{1}{2}\sum_{x=1}^{L-1}\left(f\left(\zeta^x\right) - f(\zeta)\right), \tag{A.3}$$

where $\zeta^x \in \mathcal{S}_L$ is defined by

$$\zeta^x_y := \begin{cases} \zeta_y & \text{if } y \ne x, \\ \zeta_{x-1} + \zeta_{x+1} - \zeta_x & \text{if } y = x. \end{cases}$$

When $\zeta_{x-1} = \zeta_{x+1}$, ζ displays a local extremum at x and we obtain ζ^x by flipping the corner of ξ at the coordinate x. Let U_L denote the uniform probability measure on $\mathcal{S}_L$. We can see that this Markov chain is reversible with respect to the uniform measure U_L. Therefore, U_L is the invariant probability measure for this Markov chain. The Markov chain starting with the maximal path $\overline{\wedge}$ is denoted by $(\eta_t^{\overline{\wedge}})_{t\ge 0}$. Likewise, let $(\eta_t^{U_L})_{t\ge 0}$ denote the Markov chain with generator $\mathfrak{L}$ and starting path chosen by sampling U_L. There is a one-one correspondence between this Markov chain and the symmetric simple exclusion process, for which we refer to [10, Section 2.3] for more information. Under the measure U_L, typical path $\zeta \in \mathcal{S}_L$ does not touch the x-axis, which is the following lemma.

Lemma A.1. *For all L sufficiently large, we have*

$$U_L\big(\mathcal{S}_L \setminus \mathcal{S}_L^+\big) \le e^{-\frac{1}{2}(\log L)^2}. \tag{A.4}$$

Proof. Let $\mathbf{P}$ be the law of the nearest-neighbor symmetric simple random walk on $\mathbb{Z}$, and $(S_i)_{i\in\mathbb{N}}$ be its trajectory with $S_0 = 0$. Since any trajectory of this simple random walk has the same mass, we have

$$\begin{aligned} U_L\big(\mathcal{S}_L \setminus \mathcal{S}_L^+\big) &= \mathbf{P}\big[\exists i \in [\![0, L]\!] : S_i + m \le 0 | S_L = 0\big] \\ &\le L^{\frac{1}{2}} \mathbf{P}\Big[\min_{i\in[\![0,L]\!]} S_i \le -m, S_L = 0\Big] \\ &\le 2L^{\frac{1}{2}} \mathbf{P}[S_L \le -m] \\ &\le e^{-\frac{1}{2}(\log L)^2}, \end{aligned}$$

which vanishes as L tends to infinity. The first inequality uses $\mathbf{P}[S_L = 0] \ge L^{-1/2}$, for all L sufficiently large. The second inequality uses

$$\mathbf{P}\Big[\min_{i\in[\![0,L]\!]} S_i \le -m, S_L = 0\Big] \le 2\mathbf{P}[S_L \le -m].$$

In the last inequality, we use the inequality, $\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \le n! \le e n^{n+\frac{1}{2}} e^{-n}$ for all $n \ge 1$, to obtain

$$\mathbf{P}[S_L \le -m] \le (L - m + 1) \binom{L}{\frac{L+m}{2}} 2^{-L} \le (L - m + 1) e^{-(\log L)^2}.$$

□

A.2. Compare the polymer pinning dynamics to simple exclusion process

There is a graphical construction similar to that mentioned at the beginning of Appendix A, allowing to couple the three dynamics $(\sigma_t^{\overline{\wedge},0})_{t\ge 0}$, $(\eta_t^{\overline{\wedge}})_{t\ge 0}$ and $(\eta_t^{U_L})_{t\ge 0}$ such that for all $t \ge 0$,

$$\sigma_t^{\overline{\wedge},0} \ge \eta_t^{\overline{\wedge}} \ge \eta_t^{U_L}. \tag{A.5}$$

Let $P_t^{\overline{\wedge},-}(\cdot) := \mathbb{P}(\eta_t^{\overline{\wedge}} = \cdot)$ and $P_t^{\overline{\wedge},0}(\cdot) := \mathbb{P}(\sigma_t^{\overline{\wedge},0} = \cdot)$. Intuitively, the distribution of $\sigma_t^{\overline{\wedge},0}$ is close to that of $\eta_t^{\overline{\wedge}}$ for all $t \ge 0$.

Lemma A.2. *For any given $\epsilon > 0$ and all L sufficiently large, we have*

$$\sup_{0\le t\le t_\delta} \big\| P_t^{\overline{\wedge},0} - P_t^{\overline{\wedge},-} \big\|_{\mathrm{TV}} \le \epsilon. \tag{A.6}$$

Proof. By Proposition 4.1 and the monotonicity in (A.5), we obtain

$$\begin{aligned} \sup_{0\le t\le t_\delta} \big\| P_t^{\overline{\wedge},0} - P_t^{\overline{\wedge},-} \big\|_{\mathrm{TV}} &\le \mathbb{P}\big[\exists t \in [0, t_\delta] : \sigma_t^{\overline{\wedge},0} \ne \eta_t^{\overline{\wedge}}\big] \\ &\le \mathbb{P}\Big[\exists t \in [0, t_\delta] : \min_{x\in[\![0,L]\!]} \eta_t^{\overline{\wedge}}(x) \le 0\Big] \\ &\le \mathbb{P}\Big[\exists t \in [0, t_\delta] : \min_{x\in[\![0,L]\!]} \eta_t^{U_L}(x) \le 0\Big]. \end{aligned} \tag{A.7}$$

The second inequality is based on the fact that in the coupling if $\sigma_t^{\overline{\wedge},0} \neq \eta_t^{\overline{\wedge}}$, there must exist $x \in [\![0, L]\!]$ satisfying $\eta_s^{\overline{\wedge}}(x) = 0$ for some $s \in [0, t]$. The third inequality uses the monotonicity of the dynamics, *i.e.* $\eta_t^{\overline{\wedge}} \geq \eta_t^{U_L}$ for all $t \geq 0$. The last term in (A.7) vanishes as L tends to infinity, which follows exactly as that in (4.33) of Lemma 4.8, using occupation time (4.30), strong Markov property and Lemma A.1. □

Furthermore, by [10, Theorem 2.4], for any given $\epsilon > 0$ and $t \geq t_{\delta/2}$, if L is sufficiently large, we have

$$\| P_t^{\overline{\wedge},-} - U_L \|_{\mathrm{TV}} \leq \epsilon. \tag{A.8}$$

Then we use the information of U_L to give an upper bound for the highest point of $\sigma_t^{\overline{\wedge},0}$.

Proof of Lemma 4.9. By triangle inequality, Lemma A.2 and (A.8), for $t \in [t_{\delta/2}, t_\delta]$, if L is sufficiently large, we have

$$\| P_t^{\overline{\wedge},0} - U_L \|_{\mathrm{TV}} \leq 2\epsilon. \tag{A.9}$$

By (A.9), for every $t \in [t_{\delta/2}, t_\delta]$ and L sufficiently large, we obtain

$$\mathbb{P}\big[\overline{H}(t) \geq 2L^{\frac{1}{2}}(\log L)^2\big] \leq U_L\Big(\sup_{x \in [\![0,L]\!]} \zeta_x \geq 2L^{\frac{1}{2}}(\log L)^2, \zeta \in \mathcal{S}_L\Big) + \| P_t^{\overline{\wedge},0} - U_L \|_{\mathrm{TV}} \leq 3\epsilon,$$

where the first term in the right hand side vanishes as L tends to infinity, whose proof is the same as Lemma A.1. Since $\epsilon > 0$ is arbitrary, we finish the proof. □

Appendix B: Spin system

To deduce Proposition 5.1 from [14, Theorem 1.1], we construct a monotone system $\langle \Omega_L^*, S, V_L, \mu_L^* \rangle$ which is the same as the Glauber dynamics of the polymer pinning model.

For $(x, z) \in \mathbb{N}^2$, a square with four vertices $\{(x-1, z), (x+1, z), (x, z-1), (x, z+1)\}$ is denoted as $Sq(x, z)$. Recalling Θ defined in (2.1), let $S := \{\oplus, \ominus\}$ denote the spins, and $V_L := \{Sq(x, z) : \forall (x, z) \in \Theta\}$ denote the set of all sites, which consists of all green or white squares shown in Figure 5. Each square of V_L is endowed with $\oplus$ or $\ominus$. Moreover, we give a natural order for the spins, say, $\ominus \leq \oplus$. For any given $\xi \in \Omega_L$, every square $Sq(x, z)$ lying under the path ξ is endowed with $\oplus$, while every square $Sq(x, z)$ lying above ξ is endowed with $\ominus$. This spin configuration is denoted as ξ^*. For $\xi, \xi' \in \Omega_L$, $\xi \leq \xi'$ if and only if $\xi^* \leq \xi'^*$. Let $\Omega_L^* := \{\xi^*, \xi \in \Omega_L\}$ and $\mu_L^*(\xi^*) := \mu(\xi)$.

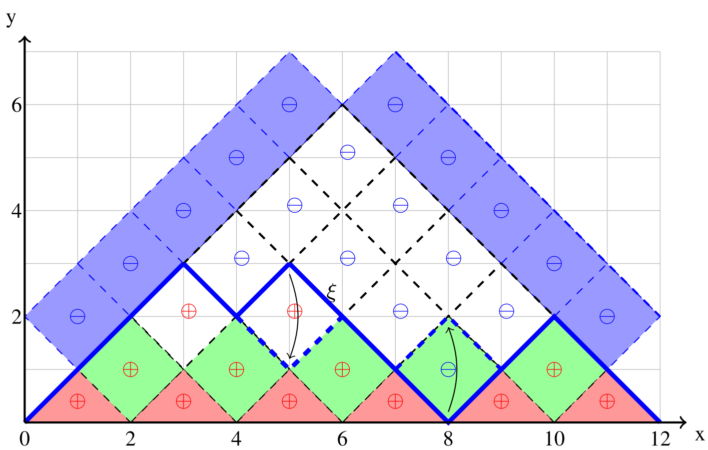

Fig. 5. An example shows the equivalence between the polymer pinning model and the spin system with $L = 12$. The blue path ξ is an element of Ω_L. This configuration in the spin system is denoted as ξ^*, and its probability measure is $\mu(\xi)$. The corner at $x = 8$ of thick blue path ξ flips up with rate $1/(1+\lambda)$ to the dashed blue corner, while the spin $\ominus$ at the green square centered at $(8, 1)$ flips to $\oplus$ with rate $1/(1+\lambda)$. The corner at $x = 5$ of thick blue path ξ flips down with rate $1/2$ to the dashed blue corner, while the spin $\oplus$ at the white square centered at $(5, 2)$ flips to $\ominus$ with rate $1/2$. Note that not all the correspondence between the flipping of the corners of ξ and that of the spins of ξ^* are shown in the picture.

For convenience of describing the Glauber dynamics of spin system, we introduce two fixed boundary conditions. We assign a negative spin $\ominus$ to each square $Sq(x,z)$ where

$$\{(x,z) : x \in [\![1, L/2-1]\!] \cup [\![L/2+1, L-1]\!], z = L/2+1-|x-L/2|\}.$$

These are the blue squares shown in Figure 5. In addition, we also introduce a positive boundary condition. A triangle with three vertices $\{(x-1,z),(x+1,z),(x,z+1)\}$ is denoted as $\mathrm{Tr}(x,z)$ for $(x,z) \in \mathbb{N}^2$. We assign a positive spin $\oplus$ to each triangle $\mathrm{Tr}(x,0)$ for all $x \in [\![1, L-1]\!] \setminus 2\mathbb{N}$. These are the red triangles shown in Figure 5. We say that two spins are neighbors if the squares or triangles on which they lie share an edge. We use the same exponential clocks and uniform coins $\mathcal{T}^\uparrow, \mathcal{T}^\downarrow, \mathcal{U}^\uparrow$, and $\mathcal{U}^\downarrow$ define in Section 2.1 to describe the dynamics of the spin system.

Given $\mathcal{T}^\uparrow, \mathcal{T}^\downarrow, \mathcal{U}^\uparrow$ and $\mathcal{U}^\downarrow$, we construct, in a deterministic way, $(\sigma_t^{\xi^*})_{t\geq 0}$ the Glauber dynamics of the spin system starting with ξ^* with parameter λ. The trajectory $(\sigma_t^{\xi^*})_{t\geq 0}$ is càdlàg with $\sigma_0^{\xi^*} = \xi^*$ and is constant in the intervals, where the clock processes are silent.

When the clock process $\mathcal{T}^\uparrow_{(x,z)}$ rings at time $t = \mathcal{T}^\uparrow_{(x,z)}(n)$ for $n \geq 1$, we update the configuration $\sigma_{t-}^{\xi^*}$ as follows:

- if the spin in the square $Sq(x,z)$ is $\ominus$, and has two neighbors with $\oplus$ spins, and $z=1$, and $\mathcal{U}^\uparrow_{(x,z)}(n) \leq \frac{1}{1+\lambda}$, we let the spin in the square $Sq(x,z)$ change to $\oplus$ at time t, and the other spins remain unchanged;
- if the spin in the square $Sq(x,z)$ is $\ominus$, and has two neighbors with $\oplus$ spins, and $z \geq 2$, and $\mathcal{U}^\uparrow_{(x,z)}(n) \leq 1/2$, we let the spin in the square $Sq(x,z)$ change to $\oplus$.

If these two conditions aforementioned are not satisfied, we do nothing.

When the clock process $\mathcal{T}^\downarrow_{(x,z)}$ rings at time $t = \mathcal{T}^\downarrow_{(x,z)}(n)$ for $n \geq 1$, we update the configuration $\sigma_{t-}^{\xi^*}$ as follows:

- if the spin in the square $Sq(x,z)$ is $\oplus$, and has two neighbors with $\ominus$ spins, and $z=1$, and $\mathcal{U}^\downarrow_{(x,z)}(n) \leq \frac{\lambda}{1+\lambda}$, we let the spin in the square $Sq(x,z)$ change to $\ominus$ at time t, and the other spins remain unchanged;
- if the spin in the square $Sq(x,z)$ is $\oplus$, and has two neighbors with $\ominus$ spins, and $z \geq 2$, and $\mathcal{U}^\downarrow_{(x,z)}(n) \leq 1/2$, we let the spin in the square $Sq(x,z)$ change to $\ominus$ at time t, and the other spins remain unchanged.

If these two conditions aforementioned are not satisfied, we do nothing.

We can see that $\langle \Omega^*, S, V, \mu^* \rangle$ is a monotone system in the sense of [14, Section 1.1], whose Glauber dynamics is the same as that of the polymer pinning model.

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