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MIXING TIME FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS IN A RANDOM ENVIRONMENT

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We consider the simple exclusion process in the integer segment $[\![1, N]\!]$ with $k \le N/2$ particles and spatially inhomogenous jumping rates. A particle at site $x \in [\![1, N]\!]$ jumps to site $x - 1$ (if $x \ge 2$) at rate $1 - \omega_x$ and to site $x + 1$ (if $x \le N - 1$) at rate ω_x if the target site is not occupied. The sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is chosen by IID sampling from a probability law whose support is bounded away from zero and one (in other words the random environment satisfies the uniform ellipticity condition). We further assume $\mathbb{E}[\log \rho_1] < 0$ where $\rho_1 := (1-\omega_1)/\omega_1$, which implies that our particles have a tendency to move to the right. We prove that the mixing time of the exclusion process in this setup grows like a power of N. More precisely, for the exclusion process with $N^{\beta+o(1)}$ particles where $\beta \in [0, 1]$, we have in the large N asymptotic

$$N^{\max(1,\frac{1}{\lambda},\beta+\frac{1}{2\lambda})+o(1)} \le t_{\mathrm{mix}}^{N,k} \le N^{C+o(1)},$$

where $\lambda > 0$ is such that $\mathbb{E}[\rho_1^\lambda] = 1$ ($\lambda = \infty$ if the equation has no positive root) and C is a constant, which depends on the distribution of ω. We conjecture that our lower bound is sharp up to subpolynomial correction.

1. Introduction.

1.1. *Overview.* From the viewpoint of probability and statistical mechanics, the simple exclusion process is one of the simplest interacting particle systems. It is a reasonable toy model to describe the relaxation of a low density gas and we refer to [28], Chapter VIII.6, for a historical introduction. Its relaxation to equilibrium has been the object of extensive study under a variety of perspective: Hydrodynamic limits [21, 33, 34], Relaxation Time [6, 32] log-Sobolev inequalites [40] and Mixing Time [3, 29] (the list of references is very far from exhaustive).

All the above mentioned works are concerned with the exclusion in an *homogeneous* medium and a small modification of this setup can lead to a drastic change of the pattern of relaxation; see, for instance, [10, 11] (and references therein) for the phenomenology induced by the change of the jump rate on a single bond. The *disordered* setup, where the jump rate of the particles is random and varies in space fostered interest only more recently; see, for instance, [7, 9, 35].

In the present paper, we are interested in the case of IID site disorder on a one-dimensional segment, in particular, in the case where the local drift felt by particles has a nonconstant sign. For the system to reach equilibrium, individual particles need to travel on macroscopic distances and in particular have to fight against drift in some regions. This phenomenon, also presents in the case of the random walk in a random environment (RWRE) [12, 19], induces a slower mixing than in the constant nonzero bias case, as was proved in [35]. Our objective is to quantify further this slowdown of the mixing time.

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In order to estimate the mixing time of the disordered exclusion process, we need to understand in detail how these regions with unfavorable drift—which we refer to as *traps*—affect the pattern of relaxation to equilibrium. We make two important steps toward this objective:

- We prove that the mixing time grows at most like a power of N (the upper bound we prove displays a nonoptimal exponent).
- We obtain a lower bound on the mixing time, which we conjecture to be optimal, and which allows to identify, depending on the parameters of the system, which is the main factor that slows down the mixing.

More precisely, our proof of the lower bound shows that the mixing time can be bounded from below by three different mechanisms:

(i) Particles cannot move faster than ballistically, so that the mixing time is at least of order N, which is the length of the system.

(ii) The particles may remain trapped in potential wells, which are created by the environment, so that the mixing time is at least of order $e^{\Delta V}$ where ΔV is the depth of the worst potential well in the system (we refer to (13) for the definition of the potential and to (58) for that of ΔV).

(iii) The potential wells also limit the flow of particles through the system, which is at most of order $e^{-\Delta V/2}$. For this last reason, the mixing time is at least of order $ke^{\Delta V/2}$ when k is the number of particles in the system. We refer to Figure 7 for an intuitive argument.

While the first two limitations (i) and (ii) follow from early studies of one-dimensional random walk in a random environment. More precisely, the introduction of the potential V is due to Solomon [37], and the potential trap approach has been used to determine the limiting behavior [19] and the mixing time [12] of the RWRE. The third limitation is specific to systems with many particles, and to our knowledge, had not been identified so far. It creates a third phase in the conjectured mixing time diagram (see Figure 4).

1.2. *The exclusion process in a random environment.* Let us introduce formally the random process whose study is the object of this paper. The exclusion process on the segment $[\![1, N]\!]$ with k particles and $1 \le k \le N/2$ is a Markov process that can informally be described as follows (we refer to Figure 1 for a graphical explanation):

(A) Each site is occupied by at most one particle (we refer to this constraint as *the exclusion rule)*. Therefore, at all times there are k occupied sites and $N-k$ empty sites.

(B) Each of the k particles performs a random walk on the segment, independently of the others, except that any jump that violates the exclusion rule is canceled.

More precisely, we want to consider the case of the exclusion process in a *random environment* where the jump rates of the particles are specified by sampling an IID sequence of random variables $\omega = (\omega_x)_{x\in\mathbb{Z}}$, and the transition rates are given by

$$(1)\qquad \begin{cases} q_N^\omega(x, x+1) = \omega_x \mathbf{1}_{\{x\le N-1\}}, \\ q_N^\omega(x, x-1) = (1-\omega_x)\mathbf{1}_{\{x\ge 2\}}, \\ q_N^\omega(x, y) = 0 & \text{if } y \notin \{x-1, x+1\}. \end{cases}$$

The random walk with transitions q_N^ω, which corresponds to the case $k=1$ is an extensively studied process, usually referred to as Random Walk in a Random Environment (RWRE). The RWRE on the full line $\mathbb{Z}$ was first studied by Solomon in [37] who established a criterion for recurrence/transience. The limit law of the random walk in a random environment is studied by Kesten et al. in [19] when the random walk is transient, and by Sinai in [36] when the

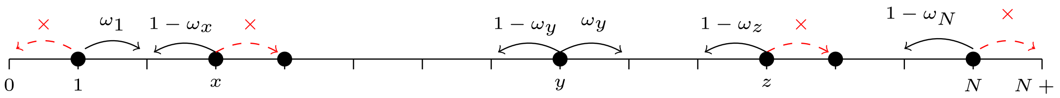

FIG. 1. *A graphical representation of the simple exclusion process in the segment* $[\![1, N]\!]$ *and environment* $\omega = (\omega_x)_{x\in\mathbb{Z}}$: *a bold circle represents a particle, and the number above every arrow represents the jump rate while a red "×" represents a nonadmissible jump.*

random walk is recurrent (we refer to [38, 41] for complete introductions to this research field).

We are interested in the following quantitative question: How long does the system need to relax to equilibrium, forgetting the information of its initial configuration in the sense of total variation distance? More precisely, we are interested in the asymptotic in the limit when $k, N \to \infty$ of this *total variation mixing time*. This question has been extensively studied in the case where the sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is constant, which we refer to as the *homogeneous environment* case:

(1) When $\omega_x \equiv \frac{1}{2}$, Wilson in [39] showed that the system takes time of order $N^2 \log \min(k, N-k)$. Later one of the authors of the present manuscript [24] proved that the lower bound in [39] is sharp.

(2) When $\omega_x \equiv p \neq \frac{1}{2}$, Benjamini et al. in [3] proved that the system takes time of order N. In [22], Labbé and one of the authors provided the exact constant.

(3) The case $\omega_x \equiv p_N = \frac{1}{2} + \varepsilon_N$ with $\lim_{N\to\infty} \varepsilon_N = 0$ is studied by Levin and Peres in [26] and also in [23].

From the results mentioned above, for homogeneous environments the system takes time at least of order N and at most of order $N^2 \log N$ to relax to equilibrium. However, when the sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is chosen by independently sampling a nondegenerate common law, the system can exhibit a very different behavior because the random environment can create wells of potential, which trap particles (see equation (13) below for a definition of the potential associated to ω).

Gantert and Kochler have studied the mixing time problem when $k = 1$ (and transient environment) in [12] for random environments and identified the mixing time, which is related to the depth of the deepest trap and may be much larger than $N^2 \log N$. Schmid [35] studied the question in the case of a positive density of particles, when the environment is ballistic to the right, (i.e., when the random walk is transient with positive speed). More precisely, he showed when $\operatorname{ess\,inf} \omega_1 = 1/2$ (i.e., the local drift $2\omega_x - 1$ is not bounded from below uniformly away from zero), the order of magnitude of the mixing time is strictly larger than N; and when $\operatorname{ess\,inf} \omega_1 < 1/2$ the mixing time is larger than $N^{1+\delta}$ for some $\delta > 0$, which depends on $\mathbb{P}$.

In our study, we focus on the case of random environments, which are such that the random walk is transient (the case of recurrent environment is quite different and should be considered separately). In that setup, the results in [35] leave several questions open, among which the following ones:

(A) Is the mixing time always bounded from above by a power of N?

(B) If this is the case, for the exclusion process with $k_N = cN^\beta \le N/2$ particles and $\beta \in [0, 1]$, can one identify an exponent $\nu = \nu(\mathbb{P}, \beta) > 0$ (depending on β and the distribution), which is such that the mixing time is of order N^ν?

We provide a positive answer to question (A) by proving an upper bound on the mixing time, which grows like a power of N. This upper bound is achieved by using a censoring procedure, which allows to transport particles one by one to their equilibrium positions. Concerning question (B), we provide a new lower bound on the mixing time, which we believe to be optimal and provide a conjecture concerning the value of ν. The bound is based on an analysis of the effect that the deepest trap has on the flow of particles through the system. Significant technical obstacles prevented us from obtaining a matching upper bound.

2. Model and result.

2.1. *An introduction to random walk in a random environment ω.* Let us recall the definition for random walk in a random environment. Given $\omega = (\omega_x)_{x\in\mathbb{Z}}$, a sequence with values in $(0, 1)$, the random walk in the environment ω is the continuous time Markov chain on $\mathbb{Z}$ whose transition rates are given by

$$\begin{cases} q^{\omega}(x, x+1) = \omega_x, \\ q^{\omega}(x, x-1) = 1 - \omega_x, \\ q^{\omega}(x, y) = 0 & \text{if } |x-y| \neq 1. \end{cases} \tag{2}$$

We let $(X_t)_{t\geq 0}$ denote the random walk in environment ω and initial condition 0 (we let Q^{ω} denote the corresponding law). This process has been extensively studied in the case where $\omega = (\omega_x)_{x\in\mathbb{Z}}$ is (the fixed realization of) a sequence of IID random variables (we will use $\mathbb{P}$ and $\mathbb{E}$ to denote the associated law and expectation, respectively), and we refer to [38, 41] for classical reviews.

Simple criteria have been derived on the distribution of ω as necessary and/or sufficient conditions for recurrence/transience, ballisticity etc. Even though most of the results are valid in a more general setup, for the sake of simplicity let us assume in the discussion that the variables $(\omega_x)_{x\in\mathbb{Z}}$ are bounded away from 0 and 1, that is, for some $\alpha \in (0, 1/2)$ we have

$$\mathbb{P}\big(\omega_1 \in [\alpha, 1-\alpha]\big) = 1. \tag{3}$$

Setting $\rho_x := (1-\omega_x)/\omega_x$, it has been proved in [37] that

$$\begin{cases} \mathbb{E}[\log \rho_1] = 0 \Rightarrow X_t \text{ is recurrent under } Q^{\omega}, \mathbb{P}\text{-a.s.}, \\ \mathbb{E}[\log \rho_1] \neq 0 \Rightarrow X_t \text{ is transient under } Q^{\omega}, \mathbb{P}\text{-a.s.} \end{cases} \tag{4}$$

More precisely in the second case we have with probability one $\lim_{t\to\infty} X_t = \infty$ (resp., $-\infty$) if $\mathbb{E}[\log \rho_1] < 0$ (resp., $\mathbb{E}[\log \rho_1] > 0$).

When transience holds, the rate at which X_t goes to infinity has also been identified in [19]. It can be expressed in terms of a simple parameter of the distribution $\mathbb{P}$ of ω, yielding in particular a necessary and sufficient condition for ballisticity. Let us assume that $\mathbb{E}[\log \rho_1] < 0$, and set

$$\lambda = \lambda_{\mathbb{P}} := \inf\{s > 0, \mathbb{E}[\rho_1^s] \geq 1\} \in (0, \infty]. \tag{5}$$

It has been proved in [19] that if $\lambda > 1$ then

$$\lim_{t\to\infty} \frac{X_t}{t} = \frac{1-\mathbb{E}[\rho_1]}{1+\mathbb{E}[\rho_1]}, \tag{6}$$

and that if $\lambda \in (0, 1]$ then

$$\lim_{t\to\infty} \frac{\log(X_t)}{\log t} = \lambda. \tag{7}$$

2.2. *The simple exclusion process in an environment ω.*

2.2.1. *Definition.* Given a sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ taking values in $(0, 1)$, $N \geq 2$ and $1 \leq k \leq N-1$, the simple exclusion process in a random environment on the line segment $[\![1, N]\!]$

(we use the notation $\llbracket a, b\rrbracket := [a, b] \cap \mathbb{Z}$) with k particles is a Markov process on

$$\Omega_{N,k} := \left\{ \xi \in \{0, 1\}^N : \sum_{x=1}^{N} \xi(x) = k \right\}. \tag{8}$$

The 1's are denoting particles while 0's correspond to empty sites. It can be informally described as follows: each of the k particles performs independently a random walk with transitions given by q^ω in (2), with the constraints that particles must remain in the segment and each site can be occupied by at most one particle. All transitions that would violate this constraint (i.e., a particle tries to jump either on sites 0, $N + 1$ or an already occupied site) are canceled.

More formally, we let $\xi^{x,y}$ be the configuration obtained by swapping the values of ξ at sites x and y of the configuration ξ, defined by

$$\forall z \in \llbracket 1, N\rrbracket, \quad \xi^{x,y}(z) = \xi(z)\mathbf{1}_{\llbracket 1,N\rrbracket\setminus\{x,y\}} + \xi(x)\mathbf{1}_{\{y\}} + \xi(y)\mathbf{1}_{\{x\}}. \tag{9}$$

The simple exclusion process in environment ω is the Markov process with transition rates given by

$$\begin{aligned} & r^\omega(\xi, \xi^{x,x+1}) := \begin{cases} \omega_x & \text{if } \xi(x) = 1 \text{ and } \xi(x+1) = 0, \\ 1 - \omega_{x+1} & \text{if } \xi(x+1) = 1 \text{ and } \xi(x) = 0 \end{cases} \quad \text{for } x \in \llbracket 1, N-1\rrbracket, \\ & r^\omega(\xi, \xi') := 0 \quad \text{in all other cases.} \end{aligned} \tag{10}$$

Equivalently, the generator of the process is defined for $f : \Omega_{N,k} \to \mathbb{R}$ by

$$\mathcal{L}^\omega_{N,k}(f)(\xi) := \sum_{x=1}^{N-1} r^\omega(\xi, \xi^{x,x+1})\left[f(\xi^{x,x+1}) - f(\xi)\right]. \tag{11}$$

Since $\omega_x \in (0, 1)$ for all x, the chain is ergodic and reversible. In order to give a simple compact expression for the equilibrium measure, let us introduce the random potential $V^\omega : \mathbb{N} \to \mathbb{R}$ defined as follows: $V^\omega(1) := 0$ and for $x \geq 2$,

$$V^\omega(x) := \sum_{y=2}^{x} \log\left(\frac{1 - \omega_y}{\omega_{y-1}}\right). \tag{12}$$

With a small abuse of notation, we extend V^ω to a function of $\Omega_{N,k}$. This extension is obtained by summing the value of V^ω among the positions of the particles in the configuration ξ:

$$V^\omega(\xi) := \sum_{x=1}^{N} V^\omega(x)\xi(x). \tag{13}$$

We consider the probability measure $\pi^\omega_{N,k}$ defined by

$$\pi^\omega_{N,k}(\xi) := \frac{1}{Z^\omega_{N,k}} e^{-V^\omega(\xi)} \quad \text{with } Z^\omega_{N,k} = \sum_{\xi \in \Omega_{N,k}} e^{-V^\omega(\xi)}. \tag{14}$$

It is immediate to check by inspection that $\pi^\omega_{N,k}$ satisfies the detailed balance condition for $\mathcal{L}^\omega_{N,k}$, and thus that it is the unique invariant probability measure on $\Omega_{N,k}$.

If $\xi \in \Omega_{N,k}$, we let $(\sigma^\xi_t)_{t\geq 0}$ denote the Markov chain with initial condition ξ. We provide in Section 3.2 a construction $(\sigma^\xi_t)_{t\geq 0}$ for all $\xi \in \Omega_{N,k}$ on a common probability space. We use $\mathbf{P}$ and $\mathbf{E}$ for the corresponding probability law and expectation, respectively. We let $(P_t)_{t\geq 0}$ (the dependence in ω, N, k is omitted in the notation to keep it light) denote the corresponding Markov semigroup and set $P^\xi_t := \mathbf{P}(\sigma^\xi_t \in \cdot) = P_t(\xi, \cdot)$ to be the marginal distribution of $(\sigma^\xi_t)_{t\geq 0}$ at time t.

2.2.2. *Mixing time and spectral gap.* In a standard fashion, we set the total variation-distance to equilibrium at time t to be

$$d_{N,k}^{\omega}(t) := \max_{\xi \in \Omega_{N,k}} \|P_t^{\xi} - \pi_{N,k}^{\omega}\|_{\mathrm{TV}}, \tag{15}$$

where $\|\nu_1 - \nu_2\|_{\mathrm{TV}} := \sup_{A \subset \Omega_{N,k}} |\nu_1(A) - \nu_2(A)|$ denotes the total variation between two probability measures ν_1, ν_2 on $\Omega_{N,k}$. Since the Markov chain is irreducible, we know that (cf. [27], Theorem 4.9)

$$\lim_{t \to \infty} d_{N,k}^{\omega}(t) = 0. \tag{16}$$

We are interested in quantitative aspects of the convergence (16). For this reason, we want to evaluate the mixing time and spectral gap of the chain (see [27] for a motivated and thorough introduction to these notions). For $\varepsilon \in (0, 1)$, the ε-mixing time of the chain is defined by

$$t_{\mathrm{mix}}^{N,k,\omega}(\varepsilon) := \inf\{t \geq 0 : d_{N,k}^{\omega}(t) \leq \varepsilon\}. \tag{17}$$

By convention, we simply write $t_{\mathrm{mix}}^{N,k,\omega}$ when $\varepsilon = 1/4$. The spectral gap of the chain $\mathrm{gap}_{N,k}^{\omega}$, in our context, is the smallest nonzero eigenvalue of $-\mathcal{L}_{N,k}^{\omega}$. Using reversibility and a spectral decomposition, it can be shown (see, for instance, [27], Corollary 12.7) that $\mathrm{gap}_{N,k}^{\omega}$ determines the asymptotic rate of convergence of $d_{N,k}^{\omega}$, or more precisely

$$\lim_{t \to \infty} \frac{1}{t} \log d_{N,k}^{\omega}(t) = -\mathrm{gap}_{N,k}^{\omega}. \tag{18}$$

The mixing time and spectral gap are related to one another by the following relation valid for $\varepsilon \in (0, 1/2)$ (cf. [27], Theorems 12.4 and 12.5):

$$\frac{1}{\mathrm{gap}_{N,k}^{\omega}} \log\left(\frac{1}{2\varepsilon}\right) \leq t_{\mathrm{mix}}^{N,k,\omega}(\varepsilon) \leq \frac{1}{\mathrm{gap}_{N,k}^{\omega}} \log\left(\frac{1}{\varepsilon \pi_{\min}}\right), \tag{19}$$

where

$$\pi_{\min} = \min_{\xi \in \Omega_{N,k}} \pi_{N,k}^{\omega}(\xi).$$

2.3. *Results.* The main object of the paper is the study of the exclusion process in an IID environment. On the way to our main result, we also prove bounds on the mixing time, which are valid for an arbitrary environment $(\omega_x)_{x \in \mathbb{Z}}$, which satisfies minimal assumptions. We present these results first.

2.3.1. *Universal bounds for the mixing time on the exclusion process.* We assume without loss of generality (by symmetry) that $k \leq N/2$. We prove that the mixing time grows at least linearly with the size of the system and at most exponentially. Both results are in a sense optimal (see the discussion in Section 2.5. below).

PROPOSITION 2.1. *Only assuming $\omega = (\omega_x)_{x \in \mathbb{Z}} \in (0, 1)^{\mathbb{Z}}$, for any $k \in [\![1, N/2]\!]$ and $N \geq 2$, we have*

$$t_{\mathrm{mix}}^{N,k,\omega} \geq \frac{1}{16} N. \tag{20}$$

Furthermore, if k_N is a sequence such that

$$k_N \leq N/2 \quad \text{and} \quad \lim_{N \to \infty} k_N = \infty, \tag{21}$$

we have for any $\varepsilon > 0$, for $N \geq N_0(\varepsilon)$ sufficiently large for any $\omega = (\omega_x)_{x \in \mathbb{Z}} \in (0, 1)^{\mathbb{Z}}$,

$$t_{\mathrm{mix}}^{N,k_N,\omega}(1 - \varepsilon) \geq \frac{1}{30} N. \tag{22}$$

For the upper bound, we require an assumption similar to (3), that is,

$$\forall x \in \mathbb{Z}, \quad \omega_x \in [\alpha, 1-\alpha]. \tag{23}$$

PROPOSITION 2.2. *Only assuming that the sequence $(\omega_x)_{x\in\mathbb{Z}}$ satisfies (23), for all $N \geq 2$ and all $k \in [\![1, N/2]\!]$, we have*

$$\mathrm{gap}^{\omega}_{N,k} \geq \alpha N^{-2}|\Omega_{N,k}|^{-1}\left(\frac{1-\alpha}{\alpha}\right)^{-N/2}, \tag{24}$$

and as a consequence for all $\varepsilon \in (0, 1/2)$,

$$t^{N,k,\omega}_{\mathrm{mix}}(\varepsilon) \leq \alpha^{-1}N^2|\Omega_{N,k}|\left(\frac{1-\alpha}{\alpha}\right)^{N/2}\left(\log|\Omega_{N,k}| + Nk\log\frac{1-\alpha}{\alpha} - \log\varepsilon\right). \tag{25}$$

2.3.2. *Mixing time for the exclusion process in a random environment*. Let us now introduce our main results concerning the exclusion process in a random environment. We assume that (3) holds, and that (recall (5))

$$\mathbb{E}[\log\rho_1] < 0, \qquad \lambda_{\mathbb{P}} < \infty \quad \text{and} \quad 1 \leq k \leq N/2. \tag{26}$$

Using the various symmetries of the the system (between left and right, particles and empty sites, etc.), the assumptions $\mathbb{E}[\log\rho_1] < 0$ and $1 \leq k \leq N/2$ entail almost no-loss of generality, the only case being left aside is a recurrent environment (i.e., $\mathbb{E}[\log\rho_1] = 0$). Assuming that $\mathbb{E}[\log\rho_1] < 0$, the assumption $\lambda_{\mathbb{P}} < \infty$ is equivalent to $\mathbb{P}[\omega_1 < 1/2] > 0$. In particular, it implies that the environment distribution is nontrivial. The case $\mathbb{P}[\omega_1 \geq 1/2] = 1$ has been addressed in [35] and is discussed in the next section.

In order to get a better intuition on the result, let us provide a description of the equilibrium measure. We introduce the event $\mathcal{A}_r \subset \Omega_{N,k}$ that the leftmost particle and rightmost empty site are at a distance smaller than r of their respective maximal and minimal possible values:

$$\mathcal{A}_r := \{\xi \in \Omega_{N,k} : \forall x \in [\![1, N-k-r]\!], \xi(x) = 0; \forall x \geq N-k+r, \xi(x) = 1\}. \tag{27}$$

The following result tells us that the mass of π_{N,k_N} is essentially concentrated at a finite distance of the configuration $\xi_{\max}$ with all k particles packed to the right (see (45)).

LEMMA 2.3. *Under the assumptions (23) and (26), we have*

$$\lim_{r\to\infty} \inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}} \mathbb{E}[\pi^{\omega}_{N,k}(\mathcal{A}_r)] = 1. \tag{28}$$

Our first main result is that if the environment satisfies the assumptions (3) and (26), the system relaxes to equilibrium in polynomial time. In other words, $t^{N,k,\omega}_{\mathrm{mix}}$ grows like a power of N with an explicit upper bound on the growth exponent. In order to describe our explicit bound, we need to introduce the function F, which is the log-Laplace transform of $\log\rho_1$, that is,

$$F(u) := \log\mathbb{E}[\rho_1^u]. \tag{29}$$

The assumption (3) ensures that $F(u) < \infty$ for every $u \in \mathbb{R}$. As the log-Laplace transform of a nontrivial random variable, F is a strictly convex function (as can be checked using Hölder's inequality). It satisfies $F(0) = F(\lambda) = 0$ (see Figure 2).

Since V^{ω} is, up to a small modification, a sum of IID variables with the same distribution as $\log\rho_1$, the function F is used to compute the large deviations of V^{ω}, and in particular to determine the geometry of the deepest potential wells. Given a sequence of events $(A_N)_{N\geq 1}$,

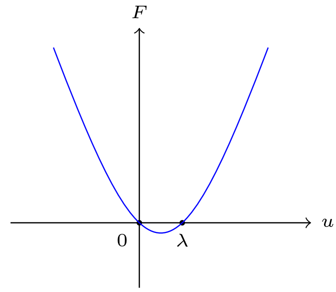

FIG. 2. *A graphical description of the function $F(u)$ with only two zeros at $u = 0$ and $u = \lambda$.*

we say that A_N holds *with high probability* (which we sometimes abbreviate as w.h.p.) if $\lim_{N\to\infty} \mathbb{P}[A_N] = 1$. Given a sequence $(B_{N,k})_{N\geq 1, k\in[\![1,N/2]\!]}$, we say that $B_{N,k}$ holds *with high probability* if

$$\lim_{N\to\infty} \inf_{k\in[\![1,N/2]\!]} \mathbb{P}[B_{N,k}] = 1.$$

We are now ready to state the result.

THEOREM 2.4. *Under the assumptions* (3) *and* (26), *then with high probability we have*

$$t_{\mathrm{mix}}^{N,k,\omega} \leq 80kN\alpha^{-1}\left(\frac{3u_0+2}{|F(u_0)|}\log N\right)^4 N^{\frac{3u_0+2}{|F(u_0)|}(2\log\frac{1-\alpha}{\alpha}+4\log 4-3\log 3)}, \tag{30}$$

where u_0 is the point at which F attains its minimum.

Our second result provides a lower bound for the mixing time, which depends both on N and k.

THEOREM 2.5. *Under the assumptions* (3) *and* (26), *there exists a positive constant $c(\alpha, \mathbb{P})$ such that w.h.p. we have*

$$t_{\mathrm{mix}}^{N,k,\omega} \geq c \max\{N, N^{\frac{1}{\lambda}}(\log N)^{-\frac{2}{\lambda}}, kN^{\frac{1}{2\lambda}}(\log N)^{-2(1+\frac{1}{\lambda})}\}. \tag{31}$$

2.4. *Comments on the uniform ellipticity assumption and possible extensions of our method.* As mentioned earlier, the assumption of uniform ellipticity of the environment (3) has been taken to make our life simpler, and to make the some of the arguments easier to expose. For instance, Proposition 2.2 which is used in the proof of Theorem 2.4 would need to be replaced by a more intricate statement if one allows for ω_x taking values in $(0, 1]$. On the other hand, we believe that our results still hold provided if only (26) is satisfied. This assumption in particular implies that $\mathbb{E}[\rho_1^u]$ is finite for some $u > 0$ but one may have $\mathbb{E}[\log \rho_1] = -\infty$. More precisely, if only (26) holds, one should have $t_{\mathrm{mix}}^{N,k,\omega} \leq N^C$ for some C depending on the distribution, and Theorem 2.5 should still holds.

The proof of Theorem 2.5 uses the ellipticity assumption only marginally, and generalizing our proof does not present much difficulty (apart from the hassle of adapting the definition of V^ω in the case when the value 1 is allowed). Ellipticity plays a more substancial role in the proof of Theorem 2.4. Since our approach is based on the hitting time of the maximal configuration (see Section 6), one can use a comparison argument and assume that ω takes value in $[\alpha, 1]$ and consider $\widetilde{\omega} := (\omega_x \wedge (1-\gamma))_{n\in\mathbb{Z}}$ (we have $\mathbb{E}[\log \widetilde{\rho}_1] \in (-\infty, 0)$ if γ is taken sufficiently small). The potential $V^{\widetilde{\omega}}$ then behaves, up to a small correction, like a random walk with negative drift, whose increments are in L^1 and whose tail distribution at

$+\infty$ are subexponential (this last point comes from $\lambda_{\mathbb{P}} < \infty$). This is sufficient to ensure that Proposition 3.4 holds for $V^{\widetilde{\omega}}$, which is the essential point to make the proof work.

Finally, let us mention that the method presented below could in principle be adapted to prove an estimate on the mixing time for the exclusion process in a Sinai type environment, that is the case when $\mathbb{E}[\log \rho_1] = 0$ and $\mathbb{E}[(\log \rho_1)^2] < \infty$. In that case, we expect that the mixing time of the simple exclusion process in such random environment satisfies w.h.p.,

$$\exp\bigl(c(\alpha, \mathbb{P})\sqrt{N}\bigr) \le t_{\mathrm{mix}}^{N,k,\omega} \le \exp\bigl(C(\alpha, \mathbb{P})\sqrt{N}\bigr). \tag{32}$$

This is due to the fact that in that case, the potential's scaling limit is given by a Brownian motion and the deepest potential trap is of order $\sqrt{N}$. The two bounds in (32) can be obtained by adapting the methods used to derive Theorem 2.4 and Theorem 2.5, respectively.

2.5. *A short review of related results.*

2.5.1. *Mixing time for the exclusion process in a homogeneous environment.* The mixing time of the exclusion process on the line segment has been extensively studied in the case where the sequence ω is constant, that is, $\omega \equiv p$. In that case, not only the right order of magnitude has been identified for the mixing time, but also the sharp asymptotic equivalent. In the case of the exclusion with no bias, that is, $p = 1/2$ (the simple symmetric exclusion process), it was shown in [1] that the mixing time for the exclusion process on the segment is of order at least N^2 and at most $N^2(\log N)^2$. It was later established (see [39] for the lower bound and [24] for the upper bound) that if k_N satisfies (21), we have

$$t_{\mathrm{mix}}^{N,k_N}(\varepsilon) = \frac{(1 + o(1))}{\pi^2} N^2 \log k_N. \tag{33}$$

In the case where the walk presents a bias, that is, $p \neq 1/2$, it was shown in [3] that the mixing time is of order N. This result was refined in [22] by identifying the proportionality constant, showing that if k_N satisfies $\lim_{N\to\infty} k_N/N = \theta$, then

$$t_{\mathrm{mix}}^{N,k_N}(\varepsilon) = [1 + o(1)] \frac{(\sqrt{\theta} + \sqrt{1-\theta})^2}{|2p-1|} N. \tag{34}$$

The case where p is allowed to depend on N was investigated in [23, 26] where the order of magnitude and the sharp asymptotic of the mixing time were respectively determined. Note that in (33) and (34) the asymptotic behavior of $t_{\mathrm{mix}}^{N,k_N}(\varepsilon)$ does not display any dependence on ε at first order. This implies that $d_{N,k_N}(t)$ abruptly drops from 1 to 0 on the time scale $N^2 \log k_N$ and N, respectively. This phenomenon, called cutoff, is expected to hold for a large class of Markov chains; we refer to [27], Chapter 18, for an introduction.

Let us also mention that the mixing time for the one-dimensional exclusion process has also been investigated for a variety of different boundary conditions. We refer to [25] for a sharp estimate of the convergence profile to equilibrium for the periodic boundary condition in the symmetric case and to [13] (and references therein) for the study of a variety of boundary conditions, with or without bias. The case of higher dimension has also been considered (see, e.g., [29]) where the order of magnitude of the mixing time is determined up to a constant.

2.5.2. *Mixing time for the random walk in a random environment.* In [12], the case of the mixing time for a random walk in the segment with a transient random environment (which corresponds to the case $k = 1$ in the present paper) was investigated. It is shown that whenever $\lambda_{\mathbb{P}} > 1$ then

$$t_{\mathrm{mix}}^{N,1,\omega}(\varepsilon) = [1 + o(1)] N \mathbb{E}\bigl[Q^{\omega}[T_1^{\omega}]\bigr], \tag{35}$$

where T_1^ω is the first hitting time of 1 for the random walk in a random environment ω starting from 0, Q^ω is the law of the random walk defined in (2) and $\mathbb{E}$ is the expectation w.r.t. the environment (more precisely the result in [12], Theorem 1.6, concerns the lazy discrete time random walk, and for this reason displays a factor 2). When $\lambda_{\mathbb{P}} < 1$, it is shown that the mixing time is of a much larger magnitude but cutoff does not hold (for a technical reason the last statement about cutoff requires an additional nonlattice assumption on the distribution of $\log \rho_1$). More precisely, for $\lambda_{\mathbb{P}} \leq 1$ we have

$$\lim_{N\to\infty} \frac{\log t_{\text{mix}}^{N,1,\omega}(\varepsilon)}{\log N} = \frac{1}{\lambda_{\mathbb{P}}}. \tag{36}$$

The asymptotic $N^{1/\lambda_{\mathbb{P}}+o(1)}$ corresponds to the time that is required to overcome the largest potential barrier present in the system, whose height is of order $(1/\lambda)\log N$.

2.5.3. *Mixing time for the exclusion in a ballistic environment.* In [35], the mixing time $t_{\text{mix}}^{N,k_N,\omega}$ was investigated under the assumption that $0 < \liminf k_N/N \leq \limsup k_N/N < 1$ and $\lambda_{\mathbb{P}} > 1$. Three different cases are considered. The following results hold with high probability w.r.t. the environment law $\mathbb{P}$:

- When $\operatorname{ess\,inf} \omega_1 > 1/2$, it is shown that the mixing $t_{\text{mix}}^{N,k_N,\omega}$ is of order N, by a simple comparison with the case of homogeneous asymmetric environment.
- When $\operatorname{ess\,inf} \omega_1 < 1/2$, it is shown that there exists a positive δ such that the mixing time satisfies $t_{\text{mix}}^{N,k_N,\omega} \geq N^{1+\delta}$.
- When $\operatorname{ess\,inf} \omega_1 = 1/2$, it is shown that

$$\liminf_{N\to\infty} t_{\text{mix}}^{N,k_N,\omega}(\varepsilon)/N = \infty \quad \text{and} \quad t_{\text{mix}}^{N,k_N,\omega}(\varepsilon) \leq CN(\log N)^3, \tag{37}$$

  together with a quantitative lower bound if $\mathbb{P}[\omega_1 = 1/2] > 0$.

2.5.4. *Other perspectives concerning the exclusion process and random environments.* The exclusion processes with other types of random environments have also been considered in the literature. One possibility is to consider a random environment on bonds instead of sites. A particular choice, which makes the uniform measure on $\mathbb{Z}$ reversible for the random walk, is the random conductance model. In that case, the mixing property of the system strongly differs from the model considered here: the equilibrium measure is uniform on $\Omega_{N,k}$ so that there is no trapping by potential. It is expected that for a large class of environments in that case the mixing properties are very similar to that of the homogeneous system. The hydrodynamic limits of exclusion processes with bond-dependent random transition rates have been studied in [7, 16] (see also [8] for a recent work going slightly beyond the random conductance model).

The papers [5, 30] study the mixing properties of the simple exclusion process with k particles on an arbitrary conductance network, that is, an arbitrary connected graph $G = (V, E)$ with N vertices and bond dependent symmetric jump rates $\omega = (\omega_e)_{e\in E}$ with $\omega_e > 0$ for all $e \in E$. In [5], it is shown that the spectral gap associated with the process is given by that of the associated simple random walk, while in [30] the mixing time of the system is compared to that of an individual particle, showing the existence of a universal constant C such that for any $k \in [\![1, N-1]\!]$, G and ω,

$$t_{\text{mix}}^{N,k,\omega} \leq C t_{\text{mix}}^{N,1,\omega} \log N.$$

Another corpus of work has been considering the (homogeneous) exclusion process itself as a dynamical random environment, which determines the transition rates of the random walk.

The asymptotic behavior of a random walker in this setup is studied in [14, 15], and the hydrodynamic limit for the exclusion process as seen by this walker is studied in [2]. In a more general setup for the jump rates of the walker, an invariance principle about the random walk when the exclusion process starts from equilibrium is studied in [17].

2.6. *Interpretation of our results and conjectures.*

2.6.1. *Comments on Propositions 2.1 and 2.2.* The asymptotic for the mixing time for ASEP in homogeneous environment (34) shows that the lower bound of Proposition 2.1 is sharp up to a constant factor. A perhaps surprising observation is also that (22) is not true without the assumption that k_N goes to infinity, even if $1/30$ is replaced by a smaller constant. In fact given $\varepsilon \le \varepsilon_0$, one can find N and ω such that $t_{\mathrm{mix}}^{N,1,\omega}(1-\varepsilon) \le N\varepsilon$.

However, the constant in our bounds (20) and (22) are clearly not optimal. Let us state now a natural conjecture. We believe that if $\lim_{N\to\infty} k_N/N = \theta \in (0, 1/2]$, and $\omega_x \in [\alpha, 1-\alpha]$ for all $x \in \mathbb{Z}$ (with the possibility of having $\alpha = 0$) then the mixing time should be minimized in the case where the environment is homogeneous and taking an extremal value, that is either $\omega \equiv \alpha$ or $\omega \equiv 1-\alpha$. This is to say (cf. [22], Theorem 2)

$$\liminf_{N\to\infty} \frac{1}{N} t_{\mathrm{mix}}^{N,k_N}(1-\varepsilon) \ge \frac{(\sqrt{\theta}+\sqrt{1-\theta})^2}{1-2\alpha}. \tag{38}$$

One can obtain counterexamples to (38) in the zero density case by considering the case $\omega_x = 1-\alpha$ in the first half of the segment $[\![1, N]\!]$ and $\omega = \alpha$ in the second half of the segment, and k_N diverging to infinity such that $\lim_{N\to\infty} k_N/(\log N) = 0$. In that case, with some minor efforts one can show that the mixing time is asymptotically equivalent $\frac{N}{2-4\alpha}$ (which is half of the lower bound in (38)).

Proposition 2.2 can also be shown to be sharp within constant in the sense that there exists a constant C_α, and for given N and k it is always possible to construct an environment ω such that

$$\mathrm{gap}_{N,k}^{\omega} \ge e^{-C_\alpha N}. \tag{39}$$

When only assuming (23), we conjecture that the best possible lower bound on the spectral gap when $\lim_{N\to\infty} k_N/N = \theta \in (0, 1/2]$ is the following:

$$\liminf_{N\to\infty} \inf_{\omega:[\![1,N]\!]\mapsto[\alpha,1-\alpha]} \frac{\log \mathrm{gap}_{N,k_N}^{\omega}}{N} = -\frac{(1-\theta)}{2}\log\left(\frac{1-\alpha}{\alpha}\right). \tag{40}$$

This conjectured lim inf is reached asymptotically by the environment

$$\omega_x = \alpha \mathbf{1}_{\{x\le N/2\}} + (1-\alpha)\mathbf{1}_{\{x>N/2\}}. \tag{41}$$

Let us briefly justify this, and we point to Figure 3 for a graphical explanation. The system of particles within $[\![1, \lfloor N/2\rfloor]\!]$ and $[\![1+\lfloor N/2\rfloor, N]\!]$ mixes rapidly (see [22]). Using a decomposition argument (see [18]), one can reduce the study of the mixing time to that of the reduced chain, which only tracks the number of particles in each half of the segment (this is a birth and death chain). The transition rate for that chain is rather explicit: a particle moves from left to right with rate of order $(\frac{\alpha}{1-\alpha})^{N/2-a}$ where a is the current number of particles on the left half, while a particle moves from right to left with rate of order $(\frac{\alpha}{1-\alpha})^{N(1/2-\theta)+a}$. At equilibrium, one has $\theta N/2 + O(1)$ particles sitting on the right and $\theta N/2 + O(1)$ particles on the left. The spectral gap of the above birth and death chain can be shown to be of order $(\frac{\alpha}{1-\alpha})^{N(1-\theta)/2}$, which corresponds to the largest effective potential barrier encountered by particles on their way to equilibrium.

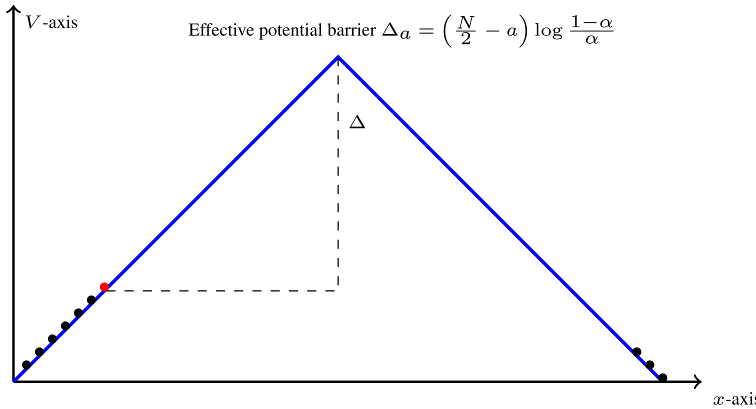

FIG. 3. *When we have $a \geq \theta N/2$ particles on the left, the rightmost of these particles has to overcome an effective potential barrier of height $\Delta_a := (\frac{N}{2} - a) \log \frac{1-\alpha}{\alpha}$. This barrier becomes higher when a approaches the equilibrium value, which is $\theta N/2$.*

2.6.2. *Comments on Theorems 2.4 and 2.5*. Our paper brings a complement to the results in [35], in the case when $\operatorname{ess\,inf} \omega_1 < 1/2$. First, it provides a complementary upper bound result, which shows that the mixing time in transient environment always scales like a power of N, even in the nonballistic case $\lambda_{\mathbb{P}} \leq 1$.

Second, it provides a more quantitative lower bound. In (31), the mixing time is bounded from below by the maximum of three quantities. Each of them corresponds to a different mechanism, which prevents the mixing time to be lower than a certain value.

- *Mass transport cannot be faster than ballistic*: What is exploited in Proposition 2.1 is that particles cannot move faster than ballistically (and this is independent of the choice of ω), so that the time required to transport the mass of particles to equilibrium has to be at least of order N. This idea is already present in [3].
- *Individual particles may be blocked by traps in the potential profile*: As soon as $\operatorname{ess\,inf} \omega_1 < 1/2$, the potential profile V^ω is nonmonotone and will display energy barriers. It is known since [19] that these energy barriers can slow down particles to subballistic speed when $\lambda_{\mathbb{P}} \leq 1$ by creating traps that will require a long time to be crossed. This is the mechanism that was used to identify the mixing time in case of a single particle in [12] (recall (36)), and it corresponds to the time needed to cross the largest trap in the potential. This yields the second term in (31).
- *Potential barrier may also create bottleneck for the flow of particles*: The third mechanism, which was partially identified in [35], is that potential barrier also limits the flow of particles throughout the system. The limitation on the flow does not correspond to the inverse of the time that a particle needs to cross the trap, but rather to the square root of this inverse. The reason for this is that when particles are flowing through the system, the particle are "filling" half of the potential well, so that the remaining potential barrier to be crossed is halved. This reasoning yields the third term in (31). We refer to Figure 7 for a graphic illustration.

We believe that the three mechanisms described above are the only limiting factors to mixing, and thus that the lower bound given in Theorem 2.5 is sharp as far as the exponent is concerned. Let us formulate this as a conjecture. Let us assume that k_N satisfies

$$\lim_{N\to\infty} \frac{\log k_N}{\log N} = \beta,$$

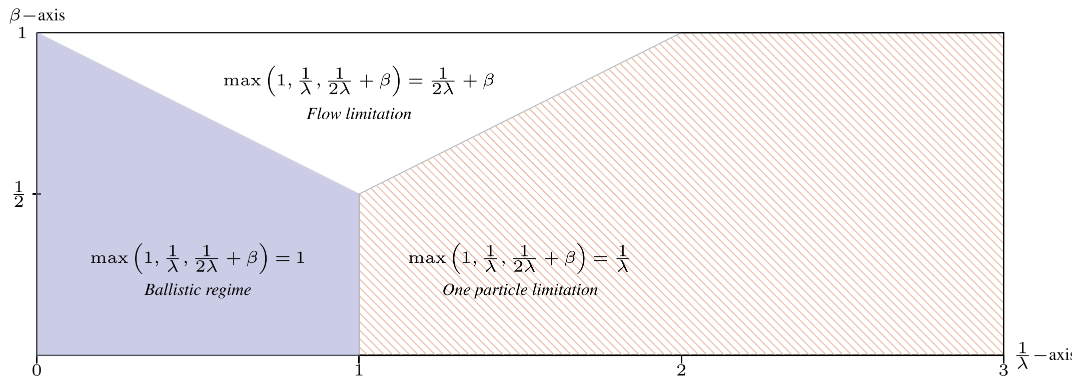

FIG. 4. *The phase diagram for the exponent of the mixing time (the lower bound is proved rigorously and the upper bound is only conjectured). The transition between the blue and red (hatched) regions of the diagram corresponds to the transition of the RWRE from the ballistic phase to the transient-with-zero-speed phase. A third phase represented by the white region appears when one considers a large number of particles; in this phase, the main limitation to mixing is the flow of particle through the deepest trap.*

and then we should have the following convergence w.h.p.:

$$\lim_{N\to\infty}\frac{\log t_{\mathrm{mix}}^{N,k_N}}{\log N}=\max\left(1,\frac{1}{\lambda},\frac{1}{2\lambda}+\beta\right). \tag{42}$$

We refer to Figure 4 for the phase diagram concerning the conjectured exponent of the mixing time.

In particular, this means that when $\beta\leq 1/(2\lambda)$ then the mixing time of the exclusion process on the segment coincides (as far as the exponent is concerned) with that of the random walk in the segment.

Since identifying the order of magnitude of the mixing time is a very challenging task, proving cutoff type results for the process seems currently out of reach. However, the nature of the mechanism that determines the mixing time presented in Figure 4 can allow to guess whether cutoff should hold or not. When in the "ballistic regime" or in the "flow limitation regime," we believe that the system should display cutoff while in the 'one particle limitation" regime, no cutoff should hold. This is because in the latter case, the mixing time of the system is determined by a single event, which takes a lot of time (namely, the time for the last particle to exit the deepest potential trap in the system), while in the other cases, what leads to mixing (either ballistic travel or flow of all the particles through a trap) can be decomposed into a diverging number of small steps. This prediction is in line with what occurs in the case of a single particle [12].

Organization. Section 3 is devoted to some technical preliminaries including the particle description, equilibrium estimates, partial order, a graphical construction and a composed censoring inequality.

Section 4 is devoted to universal lower and upper bounds on the mixing time for all random environments, that is, the proofs of Propositions 2.1 and Proposition 2.2.

Section 5 is devoted to lower bounds on the mixing time, which is Theorem 2.5. There are three bounds to prove: one of them is a consequence of Proposition 2.1, the other two are presented as two distinct results (Proposition 5.1 and Proposition 5.2) and proved in separate subsections. The first bound relies on controlling the displacement of the leftmost particle while the other is based on a control of the particle flow.

Section 6 is concerned with the upper bound on the mixing time (Theorem 2.4). The proof is based on an application of the censoring inequality and of our upper bound from Proposition 2.2: blocking the transitions along carefully chosen edges (in a way that varies through time) we guide all particles to the right of the segment (where they are typically located at equilibrium) in polynomial time.

Comments on notation. We use $c(\alpha, \mathbb{P})$ and $C(\alpha, \mathbb{P})$ to stress that the constants c and C depend on α and the law of the random environment ω.

3. Technical preliminaries.

3.1. *Partial order on $\Omega_{N,k}$.* Given $\xi \in \Omega_{N,k}$, we define $\bar{\xi} : [\![1,k]\!] \to [\![1,N]\!]$ as an increasing function, which provides the positions of the particles of ξ from left to right:

$$\{\bar{\xi}(i) = x\} \iff \left\{\xi(x) = 1 \text{ and } \sum_{y=1}^{x} \xi(x) = i\right\}. \tag{43}$$

We introduce a natural partial order relation "$\leq$" on $\Omega_{N,k} \times \Omega_{N,k}$ as follows:

$$(\xi \leq \eta) \quad \Leftrightarrow \quad \big(\forall i \in [\![1,k]\!], \bar{\xi}(i) \leq \bar{\eta}(i)\big). \tag{44}$$

Informally, $\xi \leq \eta$ means that the particles in the configuration η are located "more to the right" than those of ξ. Let $\xi_{\max}$ and $\xi_{\min}$ denote the maximal and minimal configurations of $(\Omega_{N,k}, \text{"} \leq \text{"})$, respectively, given by

$$\xi_{\max} := \mathbf{1}_{\{N-k+1 \leq x \leq N\}} \quad \text{and} \quad \xi_{\min} := \mathbf{1}_{\{1 \leq x \leq k\}}. \tag{45}$$

This order plays a special role for our dynamic $(\sigma_t^{\xi})_{t \geq 0}$, and the next two subsections provide tools to exploit this link.

3.2. *Canonical coupling via graphical construction.* Let us present a construction of a grand coupling for the exclusion process on the segment $[\![1,N]\!]$, which has the property of conserving the order defined above.

To each site $x \in [\![1,N]\!]$, we associate an independent rate 1 Poisson clock process $(T_i^{(x)})_{i \geq 1}$ (the increments of the sequence $(T_i^{(x)})_{i \geq 1}$ are IID exponential variables of parameter 1) and an independent sequence of IID variables $(U_i^{(x)})_{i \geq 1}$ with uniform distribution on [0, 1]. These variables are independent of the environment $\omega = (\omega_x)_{x \in \mathbb{Z}}$, and the trajectory $(\sigma_t^{\xi})_{t \geq 0}$ for each ξ is a deterministic function of $(T_i^{(x)}, U_i^{(x)})_{i \geq 1, x \in [\![1,N]\!]}$. In the remainder of the paper, $\mathbf{P}$ denotes the joint law of $(T_i^{(x)}, U_i^{(x)})_{i \geq 1, x \in [\![1,N]\!]}$, and $\mathbf{E}$ denotes the corresponding expectation. Let us also introduce a natural filtration $(\mathcal{F}_t)_{t \geq 0}$ in this probability space setting

$$i_0(x,t) := \max\{i \geq 1 : T_i^{(x)} \leq t\} \tag{46}$$

with the convention that $\max \varnothing = 0$ and set

$$\mathcal{F}_t := \sigma\big(T_i^{(x)}, U_i^{(x)}, x \in \mathbb{Z}, i \leq i_0(x,t)\big). \tag{47}$$

Now, given $1 \leq k \leq N-1$ and an initial configuration $\xi \in \Omega_{N,k}$, we construct the trajectory $(\sigma_t^{\xi})_{t \geq 0}$ as follows:

(1) $(\sigma_t^{\xi})_{t \geq 0}$ is càdlàg and may change its value only at times $T_i^{(x)}$, $x \in [\![1,N]\!]$ and $i \geq 1$.
(2) We construct the trajectory starting with $\sigma_0^{\xi} = \xi$ and modifying it sequentially at the update times $(T_i^{(x)})_{i \geq 1, x \in [\![1,N]\!]}$. For instance, if $t = T_i^{(x)}$ we obtain σ_t^{ξ} from $\sigma_{t_-}^{\xi}$ as follows:

(A) If $U_i^{(x)} \leq \omega_x$, $x \leq N-1$, $\sigma_{t_-}^{\xi}(x) = 1$ and $\sigma_{t_-}^{\xi}(x+1) = 0$, then $\sigma_t^{\xi}(x+1) = 1$ and $\sigma_t^{\xi}(x) = 0$ (and $\sigma_t^{\xi}(y) = \sigma_{t_-}^{\xi}(y)$ for $y \notin \{x, x+1\}$).

(B) If $U_i^{(x)} > \omega_x$, $x \geq 2$, $\sigma_{t\_}^{\xi}(x) = 1$ and $\sigma_{t\_}^{\xi}(x-1) = 0$, then $\sigma_t^{\xi}(x-1) = 1$ and $\sigma_t^{\xi}(x) = 0$ (and $\sigma_t^{\xi}(y) = \sigma_{t\_}^{\xi}(y)$ for $y \notin \{x-1, x\}$).

(C) In all other cases, $\sigma_t^{\xi} = \sigma_{t\_}^{\xi}$.

It is elementary to check by inspection that the above construction results indeed in the Markov chain with generator $\mathcal{L}_{N,k}^{\omega}$. Note also that our process is adapted and Markov with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. In the same manner, the reader can check that it preserves the order in the following sense.

PROPOSITION 3.1. *For the coupling constructed above, we have for all* $\xi, \xi' \in \Omega_{N,k}$,

$$\xi \leq \xi' \quad \Rightarrow \quad \mathbf{P}[\forall t \geq 0, \sigma_t^{\xi} \leq \sigma_t^{\xi'}] = 1. \tag{48}$$

3.3. *Composed censoring inequality.* We are going to use a variant of the censoring inequality introduced by Peres and Winkler [31]. Let $E_N = \{\{n, n+1\} : n \in [\![1, N-1]\!]\}$ be the set of edges in $[\![1, N]\!]$, and a censoring scheme $\mathcal{C} : [0, \infty) \to \mathcal{P}(E_N)$ is a deterministic càdlàg function where $\mathcal{P}(E_N)$ is the set of all subsets of E_N.

The censored chain $(\sigma_t^{\xi,\mathcal{C}})_{t\geq 0}$ is a time inhomegenous Markov chain, with a generator obtained by canceling the transitions using edges in $\mathcal{C}(t)$,

$$\mathcal{L}_{N,k}^{\mathcal{C},t}(f)(\xi) := \sum_{x=1}^{N-1} r^{\omega}(\xi, \xi^{x,x+1}) \mathbf{1}_{\{\{x,x+1\}\notin \mathcal{C}(t)\}} [f(\xi^{x,x+1}) - f(\xi)], \tag{49}$$

where $r^{\omega}(\xi, \xi^{x,x+1})$ is defined in (10). We let $P_t^{\mathcal{C}}$ be the associated semigroup (the solution of $\partial_t P_t = P_t \mathcal{L}_{N,k}^{\mathcal{C},t}$ with initial condition given by the identity). The *censoring inequality* [31], Theorem 1, states that if one starts from an extremal initial condition, censoring delays the mixing in the sense that (recall (45))

$$\| P_t^{\mathcal{C}}(\xi_{\min}, \cdot) - \pi \|_{\mathrm{TV}} \leq \| P_t(\xi_{\min}, \cdot) - \pi \|_{\mathrm{TV}}, \tag{50}$$

and furthermore, the result provides some additional information, namely that $P_t^{\mathcal{C}}(\xi_{\min}, \cdot)$ is stochastically dominated by $P_t(\xi_{\min}, \cdot)$ (to see that the exclusion process fits the setup in [31], one uses the height function representation, and the full details are provided in [24], Section A.2), so that in particular,

$$P_t(\xi_{\min}, \xi_{\max}) \geq P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}). \tag{51}$$

This yields the following consequence.

PROPOSITION 3.2. *For any* $\xi \in \Omega_{N,k}$ *and any censoring scheme* $\mathcal{C}$, *we have*

$$P_t(\xi, \xi_{\max}) \geq P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}). \tag{52}$$

PROOF. Proposition 3.1 implies that $P_t(\xi, \xi_{\max}) \geq P_t(\xi_{\min}, \xi_{\max})$ and (51) allows to conclude. □

We consider modified censored dynamics, where on top of censoring, at fixed time, we replace the current configuration by one, which is lower for the order $\geq$ by moving some particles to the left. For the application we have in mind, we can consider that these replacements are performed deterministically (although the result would hold also for random replacements).

Let $(s_i)_{i=1}^I$ be an increasing time sequence tending to infinity and let $(Q_i)_{i=1}^I$ be a sequence of stochastic matrices on $\Omega_{N,k}$ such that for all ξ in $\Omega_{N,k}$ there exists ξ' (depending on ξ and i) such that

$$\begin{cases} \xi' \leq \xi, \\ Q_i(\xi, \xi') = 1, \\ Q_i(\xi, \xi'') = 0 \quad \text{when } \xi'' \neq \xi'. \end{cases} \tag{53}$$

The stochastic matrix Q_i is simply a mapping between configurations, which sends ξ to a lower configuration ξ' deterministically. We consider $\widetilde{P}_t$ the semigroup defined by

$$\begin{cases} \widetilde{P}_0 = \mathrm{Id}, \\ \partial_t \widetilde{P}_t = \widetilde{P}_t \mathcal{L}^{\mathcal{C},t} \quad \text{if } t \notin \{s_i\}_{i=1}^I, \\ \widetilde{P}_{s_i} = \widetilde{P}_{(s_i)_-} Q_i. \end{cases} \tag{54}$$

PROPOSITION 3.3. *For any choice of $(s_i)_{i=1}^I$, $(Q_i)_{i=1}^I$ and $\mathcal{C}$, we have for all $t \geq 0$,*

$$P_t(\xi_{\min}, \xi_{\max}) \geq \widetilde{P}_t(\xi_{\min}, \xi_{\max}). \tag{55}$$

PROOF. In view of Proposition 3.2, it is sufficient to prove that

$$P_t^{\mathcal{C}}(\xi_{\min}, \xi_{\max}) \geq \widetilde{P}_t(\xi_{\min}, \xi_{\max}).$$

We perform a graphical construction of two dynamics on the same probability space, using the same auxiliary variables $(T_i^{(x)}, U_i^{(x)})_{i\geq 1, x\in [\![1,N]\!]}$ to construct both trajectories. The two dynamics are $(\widetilde{\sigma}_t^{\min})_{t\geq 0}$—with transition probability $\widetilde{P}_t$ and initial condition $\xi_{\min}$—and $(\sigma_t^{\min,\mathcal{C}})_{t\geq 0}$—the censored dynamics with the same initial condition. The construction is as follows.

For $(\sigma_t^{\min,\mathcal{C}})_{t\geq 0}$, we use the procedure given in Section 3.2 as for $(\sigma_t^{\xi})_{t\geq 0}$ (for $\xi = \xi_{\min}$) with the following added requirement for the transitions: $\{x, x+1\} \notin \mathcal{C}(t)$ in the case (A) and $\{x, x-1\} \notin \mathcal{C}(t)$ in the case (B). For $(\widetilde{\sigma}_t^{\min})_{t\geq 0}$, we use the same procedure as for $(\sigma_t^{\min,\mathcal{C}})_{t\geq 0}$ but with the addition of new deterministic jumps in the trajectories at times $(s_i)_{i\in I}$. More precisely, if $t = s_i$, $\widetilde{\sigma}_t^{\min}$ is determined from $\widetilde{\sigma}_{t_-}^{\min}$ as the unique element of $\Omega_{N,k}$ such that

$$Q_i(\widetilde{\sigma}_{t_-}^{\min}, \widetilde{\sigma}_t^{\min}) = 1. \tag{56}$$

We have by definition $\widetilde{\sigma}_0^{\min} = \sigma_0^{\min,\mathcal{C}}$, and it can be checked by inspection that all the transitions are order preserving (this is a property of the graphical construction when $t \notin \{s_i\}_{i=1}^I$ and a consequence of (53) for the special values $t \in \{s_i\}_{i=1}^I$). □

3.4. *Equilibrium estimates*. Recalling (29) let us define

$$\kappa := F'(\lambda) = \mathbb{E}[\rho_1^\lambda \log(\rho_1)] > 0, \tag{57}$$

and set

$$\Delta V_{\max}^{\omega,N} = \max_{1\leq x\leq y\leq N} (V(y) - V(x)). \tag{58}$$

The literature on the subject of random walks in a random environment contains very sharp information concerning $\Delta V_{\max}^{\omega,N}$, and the length of the corresponding trap (see [12]). In particular, it is known under quite general assumptions that $|\Delta V_{\max}^{\omega,N} - \frac{1}{\lambda}\log N|$ displays random fluctuations of order 1 and that the corresponding traps are of a length $\frac{1}{\lambda\kappa}\log N$ at first order.

For the sake of completeness, we include a short proof of the following nonoptimal result, which is sufficient for our purpose. Set

$$q_N := \left\lceil \frac{3u_0+2}{|F(u_0)|} \log N \right\rceil, \tag{59}$$

where u_0 is the point at which F attains its minimum.

PROPOSITION 3.4. *For any fixed* $\varepsilon > 0$, *we have*

$$\lim_{N\to\infty} \mathbb{P}\left[-\left(\frac{1+\varepsilon}{\lambda}\right)\log\log N \le \Delta V_{\max}^{\omega,N} - \frac{1}{\lambda}\log N \le \frac{\varepsilon}{\lambda}\log\log N\right] = 1. \tag{60}$$

Furthermore, *we have*

$$\lim_{N\to\infty} \mathbb{P}\Big[\max_{\substack{1\le x\le y\le N\\ y-x\ge q_N}} \big(V(y)-V(x)\big) \ge -3\log N\Big] = 0. \tag{61}$$

In particular, with high probability w.r.t. the environment law $\mathbb{P}$ *we have*

$$\forall x, y \in [\![1,N]\!], \quad \{V(y)-V(x) = \Delta V_{\max}^{\omega,N}\} \quad \Rightarrow \quad \{(y-x)\le q_N\}.$$

PROOF. At the cost of an additive constant on our bounds (which we omit in the proof for readability), using our uniform ellipticity assumption we can replace $V(y)-V(x)$ in the definition of (58) by a sum of IID random variables, setting $\bar V(1)=0$ and

$$\sum_{z=x+1}^{y} \log \rho_z := \bar V(y) - \bar V(x). \tag{62}$$

By definition of λ, $M_n = (\prod_{x=1}^n (\rho_x)^\lambda)_{n\ge 1}$ is a martingale for the filtration $\mathcal{G}_n := \sigma(\omega_x, x \in [\![1,n]\!])$. Using the optional stopping theorem at $T_A := \inf\{n, M_n \ge A\}$ and using that

$$\begin{cases} A \le M_{T_A} \le A\left(\dfrac{1-\alpha}{\alpha}\right)^\lambda, \\ \lim\limits_{n\to\infty} M_n = 0, \end{cases} \tag{63}$$

we have for any A

$$\frac{1}{A}\left(\frac{\alpha}{1-\alpha}\right)^\lambda \le \mathbb{P}\left[\max_{n\ge 1} \prod_{x=1}^n (\rho_x)^\lambda \ge A\right] \le \frac{1}{A}. \tag{64}$$

The bound above can be used to obtain the upper bound on $\Delta V_{\max}^{\omega,N}$ via a union bound using translation invariance

$$\begin{aligned} &\mathbb{P}\Big[\max_{1\le x\le y\le N} \bar V(y)-\bar V(x) \ge \frac{1}{\lambda}\log N + \frac{\varepsilon}{\lambda}\log\log N\Big] \\ &\quad \le \sum_{x=1}^N \mathbb{P}\Big[\max_{y\ge x} \bar V(y) - \bar V(x) \ge \frac{1}{\lambda}\log N + \frac{\varepsilon}{\lambda}\log\log N\Big] \\ &\quad \le N\mathbb{P}\left[\max_{n\ge 1}\prod_{x=1}^n (\rho_x)^\lambda \ge N(\log N)^\varepsilon\right] \le (\log N)^{-\varepsilon}. \end{aligned} \tag{65}$$

Before proving the corresponding lower bound, let us move to the proof of (61). Again using translation invariance and union bound, it is sufficient to show that

$$\lim_{N\to\infty} N\mathbb{P}\left[\max_{n\ge q_N} \sum_{x=1}^n \log\rho_x \ge -3\log N\right] = 0. \tag{66}$$

We use Doob's maximal inequality for the martingale $e^{-nF(u_0)}\prod_{x=1}^n(\rho_x)^{u_0}$. Since $F(u_0)<0$, we have

$$\begin{aligned}(67)\qquad &\mathbb{P}\left[\max_{n\geq q_N}\prod_{x=1}^n(\rho_x)^{u_0}\geq N^{-3u_0}\right]\\ &\leq \mathbb{P}\left[\max_{n\geq 1}e^{-nF(u_0)}\prod_{x=1}^n(\rho_x)^{u_0}\geq N^{-3u_0}e^{-q_NF(u_0)}\right]\leq N^{3u_0}e^{q_NF(u_0)}\leq N^{-2}.\end{aligned}$$

This is sufficient to conclude the proof of (61). Note that as a consequence of (64) (lower bound) and (67), we have for N sufficiently large

$$(68)\qquad \mathbb{P}\left[\max_{1\leq n\leq q_N}\prod_{x=1}^n(\rho_x)^{\lambda}\geq N(\log N)^{-(1+\varepsilon)}\right]\geq\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^{\lambda}N^{-1}(\log N)^{1+\varepsilon}.$$

As a consequence of independence, we have

$$\begin{aligned}(69)\qquad &\mathbb{P}\Bigg[\forall(i,j)\in[\![1,\lfloor N/q_N\rfloor-1]\!]\times[\![1,q_N]\!]:\\ &\qquad \bar V(iq_N+j)-\bar V(iq_N)\leq\frac{\log N-(1+\varepsilon)\log\log N}{\lambda}\Bigg]\\ &\leq\left(1-\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^{\lambda}N^{-1}(\log N)^{(1+\varepsilon)}\right)^{\lfloor N/q_N\rfloor-1}\leq e^{-c(\log N)^{\varepsilon}}.\end{aligned}$$

This yields the lower bound in (60). □

PROOF OF LEMMA 2.3. We use the same argument as in the proof of the rougher bound [35], Lemma 4.1. We reproduce it here for the sake of completeness. For $\xi\in\Omega_{N,k}$, we define the positions of its leftmost particle and rightmost empty site to be respectively

$$\begin{aligned}(70)\qquad L_{N,k}(\xi)&:=\inf\{x\in[\![1,N]\!]:\xi(x)=1\},\\ R_{N,k}(\xi)&:=\sup\{x\in[\![1,N]\!]:\xi(x)=0\}.\end{aligned}$$

Then

$$\pi^{\omega}_{N,k}(\mathcal{A}_r^{\complement})\leq\pi^{\omega}_{N,k}\big(L_{N,k}(\xi)\leq N-k-r\big)+\pi^{\omega}_{N,k}\big(R_{N,k}(\xi)\geq N-k+r\big).$$

Let us bound the second term, and the first one can be treated in a symmetric manner. Moreover, we have

$$(71)\qquad \pi^{\omega}_{N,k}\big(R_{N,k}(\xi)\geq N-k+r\big)=\sum_{\substack{x\in[\![1,N-k]\!]\\ y\in[\![N-k+r,N]\!]}}\pi^{\omega}_{N,k}(L_{N,k}=x,R_{N,k}=y).$$

Furthermore, we recall that $\xi^{x,y}$, defined in (9), denotes the configuration obtained by swapping the values at sites x, y of the configuration ξ, and observe that the map $\xi\mapsto\xi^{x,y}$ is injective from $\{\xi\in\Omega_{N,k}:L_{N,k}(\xi)=x,R_{N,k}(\xi)=y\}$ to $\Omega_{N,k}$. Then we have

$$\begin{aligned}(72)\qquad \pi^{\omega}_{N,k}(L_{N,k}=x,R_{N,k}=y)&=\sum_{\{\xi:L_{N,k}(\xi)=x,R_{N,k}(\xi)=y\}}\pi^{\omega}_{N,k}(\xi^{x,y})e^{V^{\omega}(y)-V^{\omega}(x)}\\ &\leq e^{V^{\omega}(y)-V^{\omega}(x)}\leq\frac{1-\alpha}{\alpha}e^{\bar V^{\omega}(y)-\bar V^{\omega}(x)}.\end{aligned}$$

Now by the law of large numbers applied to a sum of IID variables, we have

$$\lim_{r\to\infty} \inf_{\substack{N\geq 1 \\ k\in[\![1,N/2]\!]}} \mathbb{P}\Big[\forall(x,y)\in[\![1,N-k]\!]\times[\![N-k+r,N]\!]: \bar{V}^{\omega}(y)-\bar{V}^{\omega}(x)\leq \frac{(y-x)\mathbb{E}[\log\rho_1]}{2}\Big]=1. \tag{73}$$

Moreover, since

$$\sum_{\substack{x\in[\![1,N-k]\!] \\ y\in[\![N-k+r,N]\!]}} e^{\frac{\mathbb{E}[\log\rho_1](y-x)}{2}} \leq \frac{e^{\mathbb{E}[\log\rho_1]r/2}}{(1-e^{\mathbb{E}[\log\rho_1]/2})^2}$$

we have

$$\lim_{r\to\infty} \inf_{\substack{N\geq 1 \\ k\in[\![1,\frac{N}{2}]\!]}} \mathbb{P}\Big[\pi^{\omega}_{N,k}\big(R_{N,k}(\xi)\geq N-k+r\big)\leq \frac{1-\alpha}{\alpha}\big(1-e^{\frac{\mathbb{E}[\log\rho_1]}{2}}\big)^{-2}e^{\frac{\mathbb{E}[\log\rho_1]r}{2}}\Big]=1, \tag{74}$$

which concludes the proof. □

4. Bounds for the mixing time with arbitrary environments.

4.1. *Proof of Proposition* 2.1. In this proof, we only assume that $\omega_x\in(0,1)$ for all $x\in\mathbb{Z}$. We look at the variable

$$m(\xi):=\sum_{x=1}^{N} x\xi(x).$$

Note that $m(\xi)\in[\frac{k(k+1)}{2},\frac{k(2N-k+1)}{2}]$. We assume that

$$\pi^{\omega}_{N,k}\Big(m(\xi)\geq \frac{k(N+1)}{2}\Big)\geq 1/2$$

(the other case can be treated symmetrically). Now, since at all times, each particle jumps to right with a rate which is at most one, starting from $\xi_{\min}$ (we write $\sigma^{\min}_t$ for $\sigma^{\xi_{\min}}_t$ to lighten the notation) we have

$$\mathbf{E}\big[m(\sigma^{\min}_t)\big]\leq \frac{k(k+1)}{2}+kt. \tag{75}$$

As a consequence of Markov's inequality, we have

$$\begin{aligned}\mathbf{P}\Big[m(\sigma^{\min}_t)\geq \frac{k(N+1)}{2}\Big]&=\mathbf{P}\Big[m(\sigma^{\min}_t)-\frac{k(k+1)}{2}\geq \frac{k(N-k)}{2}\Big]\\&\leq \frac{2t}{(N-k)},\end{aligned} \tag{76}$$

which is smaller than $1/4$ if $t\leq N/16$.

When the number of particles goes to infinity, we use the same kind of reasoning but adding concentration estimates for $m(\xi)$, under the equilibrium measure $\pi^{\omega}_{N,k}$ (which is denoted simply by π in this proof for readability). Let us prove that

$$\mathrm{Var}_{\pi}\big[m(\xi)\big]\leq N^2k. \tag{77}$$

To this end, we introduce the filtration $(\mathcal{G}_i)_{i=1}^{N}$ defined by $\mathcal{G}_i:=\sigma(\xi(x),x\in[\![1,i]\!])$, and consider the martingale

$$M_i:=E_{\pi}\big[m(\xi)|\mathcal{G}_i\big], \tag{78}$$

where $E_\pi[\cdot|\mathcal{G}_i]$ denotes the conditional expectation under π. We have by construction

$$\mathrm{Var}_\pi\big[m(\xi)\big] = \sum_{i=1}^{N} \mathrm{Var}(M_i - M_{i-1}). \tag{79}$$

Now, we are going to show that

$$\mathrm{Var}(M_i - M_{i-1}) \le \pi(\xi_i = 1)(N-i)^2, \tag{80}$$

which implies (77). To prove (80), we are going to show that for any $\chi \in \{0,1\}^{i-1}$ with at most $k-1$ ones and at most $N-k-1$ zeros, the quantity

$$\Delta_i(\chi) = E_\pi\big[m(\xi)|\xi_{[\![1,i-1]\!]} = \chi, \xi(i) = 0\big] - E_\pi\big[m(\xi)|\xi_{[\![1,i-1]\!]} = \chi, \xi(i) = 1\big] \tag{81}$$

satisfies

$$0 \le \Delta_i(\chi) \le N - i. \tag{82}$$

Note that we have

$$E_\pi\big[m(\xi)|\xi_{[\![1,i-1]\!]} = \chi\big] = \sum_{x=1}^{i-1} x\chi(x) + \pi^{\omega}_{[\![i,N]\!],k-\sum_{x=1}^{i-1}\chi(x)}\Bigg(\sum_{x=i}^{N} x\xi(x)\Bigg), \tag{83}$$

where if I is a segment on $\mathbb{Z}$ and $k' \le |I|$, $\pi^{\omega}_{I,k'}$ denotes the equilibrium measure for exclusion process on I with k' particles and environment ω. For this reason, it is sufficient to prove (81) for $i = 1$, and arbitrary k (not necessarily assuming $k \le N/2$). Hence, we need to prove that for $N \ge 1$ and $k \in [\![1, N-1]\!]$ we have

$$0 \le E_\pi\big[m(\xi)|\xi(1) = 0\big] - E_\pi\big[m(\xi)|\xi(1) = 1\big] \le N - 1. \tag{84}$$

To prove this, we observe that there exists a probability Π on $\Omega^2_{N,k}$ with marginals $\pi(\cdot|\xi(1) = 0)$ and $\pi(\cdot|\xi(1) = 1)$ such that

$$\Pi\Bigg(\sum_{x=1}^{N} \mathbf{1}_{\{\xi^1(x)\neq\xi^2(x)\}} = 2\Bigg) = 1 \tag{85}$$

(meaning that $\xi^1(x) = \xi^2(x)$ except at two sites, 1 and another random site). With this coupling, we have

$$E_\pi\big[m(\xi)|\xi(1) = 0\big] - E_\pi\big[m(\xi)|\xi(1) = 1\big] = \Pi\Bigg[\sum_{x=1}^{N} x\big(\xi^1(x) - \xi^2(x)\big)\Bigg],$$

which yields (84). The coupling Π can be achieved using the graphical construction: we define $(\xi^1_t)_{t\ge0}$ and $(\xi^2_t)_{t\ge0}$ starting with initial configuration $\mathbf{1}_{[\![2,k+1]\!]}$ and $\mathbf{1}_{[\![1,k]\!]}$, respectively, and evolving using the graphical construction with the edge $\{1,2\}$ censored (recall Section 3.3). The dynamic conserves the number of discrepancies and $\pi(\cdot|\xi(1) = 0)$ and $\pi(\cdot|\xi(1) = 1)$ are the respective equilibrium distributions of the marginals, so that any limit point of $\mathbf{P}[(\xi^1_t, \xi^2_t) \in \cdot]$ (existence is ensured by compactness) provides a coupling satisfying (85).

Now to see that (82) implies (80), we simply observe that, conditioned on the state of the first $i-1$ vertices of the segment, $(M_i - M_{i-1})$ can only assume two values which differ by an amount $\Delta_i(\chi)$ (cf. (81)). The corresponding conditioned variance is equal to $\Delta_i(\chi)^2$ times that of the corresponding Bernoulli variable, that is,

$$\begin{aligned}
&E_\pi\big[(M_i - M_{i-1})^2|\xi_{[\![1,i-1]\!]} = \chi\big] \\
&\quad= \pi\big(\xi(i) = 1|\xi_{[\![1,i-1]\!]} = \chi\big)\pi\big(\xi(i) = 0|\xi_{[\![1,i-1]\!]} = \chi\big)\Delta_i(\chi)^2 \\
&\quad\le \pi\big(\xi(i) = 1|\xi_{[\![1,i-1]\!]} = \chi\big)(N-i)^2.
\end{aligned} \tag{86}$$

Then we take the average with respect to $\xi_{[\![1,i-1]\!]}$ in the above inequality to conclude. Now using (77) we can assume that for any ε there exists $N_0(\varepsilon)$ such that for $N \geq N_0(\varepsilon)$ we have

$$\min\left[\pi^{\omega}_{N,k}\left(m(\xi) \leq Nk/3\right), \pi^{\omega}_{N,k}\left(m(\xi) \geq 2Nk/3\right)\right] \leq \varepsilon/2. \tag{87}$$

Let us assume that the first of these two terms is smaller (the other case is treated symmetrically). To conclude, we must show that for $t = \frac{N}{30}$ we have

$$\mathbf{P}\left(m\left(\sigma_t^{\min}\right) > Nk/3\right) \leq \varepsilon/2. \tag{88}$$

To check this, we observe that

$$m\left(\sigma_t^{\min}\right) \leq \frac{k(k+1)}{2} + \mathcal{N}_t, \tag{89}$$

where $\mathcal{N}_t$ is the total number of particle jumps to the right up to time t. Since each particle jumps at most with rate one, we have for N sufficiently large

$$\mathbf{P}[\mathcal{N}_t \geq 2kt] \leq \varepsilon/2, \tag{90}$$

which allows to conclude.

4.2. *Proof of Proposition* 2.2. For the proof of Proposition 2.2, we only assume (23) and apply the so-called flow method (see [27], Chapter 13.4). A path Γ is a sequence of configurations $(\xi_0, \dots, \xi_{|\Gamma|})$, which is such that $r^{\omega}(\xi_{i-1}, \xi_i) > 0$ for $i \in [\![1, |\Gamma|]\!]$. For any given ordered pair $(\xi, \xi') \in \Omega_{N,k} \times \Omega_{N,k}$, we assign a path $\Gamma_{\xi,\xi'}$, whose starting point is ξ and ending point is ξ'.

Using [27], Corollary 13.21, the spectral gap of the chain can be controlled by a simple quantity depending on the functional $(\xi, \xi') \mapsto \Gamma_{\xi,\xi'}$. We say that an unordered pair $e = \{\xi, \xi'\} \subset \Omega_{N,k}$ is an edge if $q(e) := \pi^{\omega}_{N,k}(\xi) r^{\omega}(\xi, \xi') > 0$ (note that by reversibility $q(e)$ does not depend on the orientation). We write $e \in \Gamma = (\xi_0, \dots, \xi_{|\Gamma|})$ if there exists $i \in [\![1, |\Gamma|]\!]$ such that $e = \{\xi_{i-1}, \xi_i\}$. As the chain is reversible, we have then (the factor $1/2$ is irrelevant but appears because we are considering unoriented edges rather than oriented ones)

$$\mathrm{gap}^{\omega}_{N,k} \geq \left(\max_e \frac{1}{2q(e)} \sum_{(\xi,\xi') \in \Omega_{Nk} \times \Omega_{N,k} : e \in \Gamma_{\xi,\xi'}} \pi^{\omega}_{N,k}(\xi) \pi^{\omega}_{N,k}(\xi') |\Gamma_{\xi,\xi'}|\right)^{-1}. \tag{91}$$

In the proof, we describe a choice for $\Gamma_{\xi,\xi'}$, which yields a relevant bound for the spectral gap. Let us fix a state $\xi^* \in \Omega_{N,k}$ that has maximal probability, which is such that

$$V^{\omega}(\xi^*) = \min_{\xi \in \Omega_{N,k}} V^{\omega}(\xi) \tag{92}$$

(we make an arbitrary choice if there are several minimizers). Now to build the path $\Gamma_{\xi,\xi'}$ we are going to build first a path from ξ to ξ^* and then one from ξ^* to ξ' and then concatenate the two.

We can thus focus on the construction of Γ_{ξ,ξ^*}. Let

$$m := d_H(\xi, \xi^*) := \frac{1}{2} \sum_{x=1}^{N} |\xi(x) - \xi^*(x)|$$

denote one-half of the Hamming distance between ξ and ξ^*. Our first step is to build a sequence $\xi^{(0)}, \dots, \xi^{(m)}$, which reduces the Hamming distance in incremental steps, that is, such that

$$\begin{cases} \xi^{(0)} = \xi \quad \text{and} \quad \xi^{(m)} = \xi^*, \\ d_H(\xi^{(i-1)}, \xi^{(i)}) = 1 \quad \text{for } i \in [\![1, m]\!], \\ d_H(\xi^{(i)}, \xi^*) = m - i \quad \text{for } i \in [\![1, m]\!]. \end{cases} \tag{93}$$

The choice we make for $\xi^{(0)}, \dots, \xi^{(m)}$ is not relevant for the result but let us fix one for the sake of clarity. Let the sequences $(x_i)_{i=1}^m$ and $(y_i)_{i=1}^m$ be defined by

$$\begin{aligned} x_i &:= \min\left\{x \in [\![1, N]\!] : \sum_{x=1}^{N} (\xi(x) - \xi^*(x))_+ = i\right\}, \\ y_i &:= \min\left\{y \in [\![1, N]\!] : \sum_{y=1}^{N} (\xi^*(y) - \xi(y))_+ = i\right\}. \end{aligned} \tag{94}$$

These sequences locate the discrepancies between ξ and ξ^*. Then we define $\xi^{(i)}$ inductively as being obtained from $\xi^{(i-1)}$ by moving the particle at x_i to y_i, which is equivalent to setting

$$\xi^{(i)} = \xi \wedge \xi^* + \sum_{j=1}^{i} \mathbf{1}_{\{y_j\}} + \sum_{j'=i+1}^{m} \mathbf{1}_{\{x_{j'}\}}.$$

Finally, our path from ξ to ξ^* is defined by concatenating paths $\Gamma^{(i)}$, $i \in [\![1, m]\!]$, linking $\xi^{(i-1)}$ to $\xi^{(i)}$. We define $\Gamma^{(i)} = (\xi_0^{(i)}, \dots, \xi_{|x_i - y_i|}^{(i)})$ as a path of minimal length $|x_i - y_i|$ linking $\xi_0^{(i)} := \xi^{(i-1)}$ to $\xi_{|x_i-y_i|}^{(i)} = \xi^{(i)}$. To define the intermediate steps, let us assume for notational simplicity (and without loss of generality) that $x_i < y_i$. Moreover, let $(z_j)_{j=1}^b$ be defined as the decreasing sequence such that (we refer to Figure 5 for a graphical description)

$$\xi^{(i-1)}|_{[\![x_i, y_i]\!]} = \mathbf{1}_{\{z_j\}_{j=1}^b}.$$

We then set $d_j := y_i - z_j$ if $j \in [\![1, b]\!]$ and $d_0 := 0$, and define $(\xi_\ell^{(i)})_{\ell=1}^{y_i - x_i}$ by setting if $d_{j-1} < \ell \le d_j$,

$$\xi_\ell^{(i)} := \xi^{(i-1)} - \mathbf{1}_{\{z_j\}} + \mathbf{1}_{\{z_j + \ell - d_{j-1}\}}. \tag{95}$$

In other words, we move the particle at site z_j ($j \ge 1$) to site z_{j-1} (with $z_0 = y_i$) starting from $j = 1$ until $j = b$.

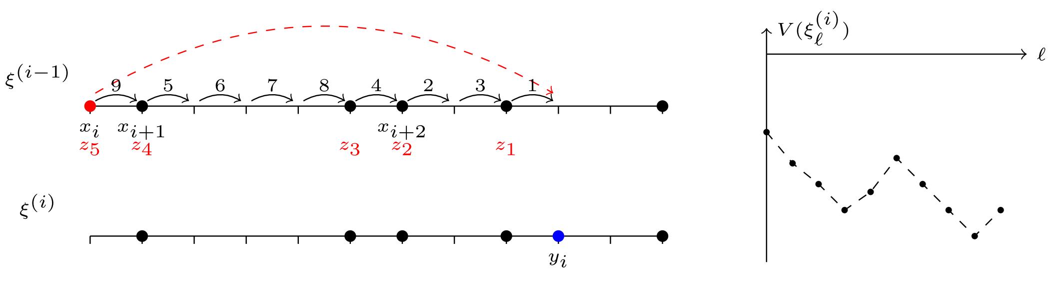

FIG. 5. *A bold circle represents a particle, and a particle at the same site for the configurations $\xi^{(i-1)}$ and $\xi^{(i)}$ is colored black. Otherwise, it is red or blue.* (L) *A graphical description of the movements of the particle at site x_i of $\xi^{(i-1)}$ to the empty site y_i and the numbers above the arrows are the relative order of the movements.* (R) *We draw the graph of $(\ell, V(\xi_\ell^{(i)}))_\ell$.*

LEMMA 4.1. *For the path collection* $(\Gamma_{\xi,\xi'})$ *constructed above, we have*

$$B := \max_e \frac{1}{2q(e)} \sum_{(\xi,\xi')\in\Omega_{Nk}\times\Omega_{N,k}:e\in\Gamma_{\xi,\xi'}} \pi^\omega_{N,k}(\xi)\pi^\omega_{N,k}(\xi')|\Gamma_{\xi,\xi'}| \le \alpha^{-1}N^2|\Omega_{N,k}|\left(\frac{1-\alpha}{\alpha}\right)^{N/2}. \tag{96}$$

Let us now conclude the proof of Proposition 2.2. By (91) and Lemma 4.1, we have

$$\mathrm{gap}^\omega_{N,k} \ge \alpha N^{-2}|\Omega_{N,k}|^{-1}\left(\frac{1-\alpha}{\alpha}\right)^{-N/2}. \tag{97}$$

Observe that

$$\max_{\xi,\xi'\in\Omega_{N,k}} \big(V^\omega(\xi) - V^\omega(\xi')\big) \le Nk\log\frac{1-\alpha}{\alpha},$$

and then

$$\min_{\xi\in\Omega_{N,k}} \pi^\omega_{N,k}(\xi) \ge |\Omega_{N,k}|^{-1}\left(\frac{1-\alpha}{\alpha}\right)^{-Nk}. \tag{98}$$

By (19), we have for $\varepsilon \in (0, 1/2)$,

$$t^{N,k,\omega}_{\mathrm{mix}}(\varepsilon) \le \alpha^{-1}N^2|\Omega_{N,k}|\left(\frac{1-\alpha}{\alpha}\right)^{N/2}\left(\log|\Omega_{N,k}| + Nk\log\frac{1-\alpha}{\alpha} - \log\varepsilon\right). \tag{99}$$

PROOF OF LEMMA 4.1. A first observation is that by construction, our paths are of length smaller than N^2. Let e be an edge and (ξ,ξ') such that $e\in\Gamma_{\xi,\xi'}$. By symmetry and taking away the factor $1/2$, we can always assume that e belongs to the first part of the path linking ξ to ξ^*. After replacing $|\Gamma_{\xi,\xi'}|$ by the upper bound and summing over all ξ', we obtain that the quantity we want to bound is exactly

$$\frac{1}{2q(e)} \sum_{(\xi,\xi')\in\Omega_{N,k}\times\Omega_{N,k}:e\in\Gamma_{\xi,\xi'}} \pi^\omega_{N,k}(\xi)\pi^\omega_{N,k}(\xi')|\Gamma_{\xi,\xi'}| \le N^2 \sum_{\xi\in\Omega_{N,k}:e\in\Gamma_{\xi,\xi^*}} \frac{\pi^\omega_{N,k}(\xi)}{q(e)}. \tag{100}$$

Now let $\chi_0(e,\xi)$ denote the first end of e, which is visited by the path going from ξ to ξ^*. Now simply observing that $q(e)$ is at least α times the smallest probability $\pi^\omega_{N,k}$ of its two end points, we have

$$\frac{\pi^\omega_{N,k}(\xi)}{q(e)} \le \sup_{\xi'\in\Gamma_{\xi,\xi^*}} \alpha^{-1}e^{V(\xi')-V(\xi)}. \tag{101}$$

Hence, using the bound in the sum in (100) we obtain that

$$\log B \le \log\alpha^{-1}N^2|\Omega_{N,k}| + \sup_{\substack{\xi\in\Omega_{N,k}\\ \xi'\in\Gamma_{\xi,\xi^*}}} V(\xi') - V(\xi). \tag{102}$$

To conclude, we only need to prove that for every $\xi\in\Omega_{N,k}$ and $\xi'\in\Gamma_{\xi,\xi^*}$ we have

$$V(\xi') - V(\xi) \le \frac{N}{2}\log\left(\frac{1-\alpha}{\alpha}\right). \tag{103}$$

This follows simply by inspection from the observations below. They are consequences of the specific construction of the flow and of assumption (23).

(i) In one step of Γ_{ξ,ξ^*}, V varies at most by $\log(\frac{1-\alpha}{\alpha})$ in absolute value.

(ii) Along the sequence $(\xi^{(i)})_{i=1}^m$, $V(\xi^{(i)})$ is nonincreasing. Indeed it follows from the definition of ξ^* that $V(y_i) \le V(x_i)$.

(iii) Each concatenated path $\Gamma^{(i)}$ has a length smaller than N (hence each $\xi_\ell^{(i)}$ is within $N/2$ steps of either $\xi^{(i)}$ or $\xi^{(i-1)}$) so that we have

$$\begin{aligned}\max_{0\le\ell\le|x_i-y_i|}\big(V\big(\xi_\ell^{(i)}\big)-V(\xi)\big) &\le \max_{0\le\ell\le|x_i-y_i|}\big(V\big(\xi_\ell^{(i)}\big)-V\big(\xi^{(i)}\big)\wedge V\big(\xi^{(i-1)}\big)\big)\\ &\le \frac{N}{2}\log\frac{1-\alpha}{\alpha}.\end{aligned}\tag{104}$$

□

5. Lower bounds on the mixing time. Theorem 2.5 contains three separate lower bounds. The first one is a consequence of Proposition 2.1. In this section, we are going to prove the two remaining bounds, which are restated below as Propositions 5.1 and 5.2, respectively. The proofs of these propositions rely on the two mechanisms exposed in Section 2.6: The potential barrier created by rare fluctuations of V^ω (cf. Proposition 3.4) has the effect of trapping individual particles and slowing down the particle flow.

5.1. *A lower bound from the position of the first particle.*

PROPOSITION 5.1. *Assuming* (3), (26) *and* $k \in [\![1, N/2]\!]$, *we have with high probability w.r.t. the environment law* $\mathbb{P}$,

$$t_{\mathrm{mix}}^{N,k,\omega} \ge \big[N(\log N)^{-2}\big]^{\frac{1}{\lambda}}.\tag{105}$$

PROOF. As a consequence of Lemma 2.3, the probability of finding a particle in the first quarter of the segment at equilibrium is small. That is, we have w.h.p. with respect to the environment (recall the notation (43))

$$\pi_{N,k}^\omega\big(\bar\xi(1) \le N/4\big) \le 1/4.\tag{106}$$

We are going to compare this probability to the one obtained for the the dynamic starting with initial condition $\xi_{\min}$. Setting $\tau_z := \inf\{t \ge 0 : \bar\sigma_t^{\min}(1) = z\}$ and using (106) we have for all $t \ge 0$,

$$d_{N,k}^\omega(t) \ge \mathbf{P}\bigg[\bar\sigma_t^{\min}(1) \le \frac{N}{4}\bigg] - \pi_{N,k}^\omega\big(\bar\xi(1) \le N/4\big) \ge 3/4 - \mathbf{P}[\tau_{N/4} \le t].\tag{107}$$

In order to estimate the time required for the leftmost particle to reach site $N/4$, we identify the highest potential barrier in this part of the segment. We let $x_1 \le y_1$ be elements of $[\![1, N/4]\!]$ such that (for all choices of x_1 and y_1 if there are several possibilities)

$$V^\omega(y_1) - V^\omega(x_1) = \max_{1\le x\le y\le N/4}\big(V^\omega(y) - V^\omega(x)\big).$$

We are going to show that for all $t \ge 0$,

$$\mathbf{P}[\tau_{N/4} \le t] \le \mathbf{P}[\tau_{y_1} \le t] \le e(t+1)e^{V(x_1)-V(y_1)}.\tag{108}$$

From (108) and (107), we deduce that

$$t_{\mathrm{mix}}^{N,k,\omega} \ge \frac{1}{2e}e^{V(y_1)-V(x_1)} - 1,\tag{109}$$

and finally as a consequence of Proposition 3.4 (applied to the segment $[\![1, N/4]\!]$), we have w.h.p.,

$$V^\omega(y_1) - V^\omega(x_1) \ge \frac{1}{\lambda}\log N - \frac{2}{\lambda}\log\log N + \log 20,$$

which allows to conclude the proof. Let us now prove (108). Using the graphical construction (with an enlargement of the probability space to sample the initial condition), we can couple $\sigma_t^{\min}$ with X_t^π a random walk on the interval $[\![1, y_1]\!]$ with transitions rates given by $q_{y_1}^\omega$ (cf. (1)) and starting with an initial distribution sampled from the equilibrium measure $\pi_{y_1,1}^\omega$, in such a way that

$$\forall t \le \tau_{y_1}, \quad \bar{\sigma}_t^{\min}(1) \le X_t^\pi.$$

Setting $\widetilde{\tau}_{y_1} := \inf\{t \ge 0 : X_t^\pi = y_1\}$, we then have for all $t \ge 0$,

$$\mathbf{P}[\tau_{y_1} \le t] \le \mathbf{P}[\widetilde{\tau}_{y_1} \le t]. \tag{110}$$

We define the occupation time

$$u(t) := \int_0^t \mathbf{1}_{\{y_1\}}(X_s^\pi)\, \mathrm{d}s.$$

We have

$$\begin{aligned} \mathbf{E}[u(t+1)] &\ge \mathbf{P}[u(t+1) \ge 1] \\ &\ge \mathbf{P}[\widetilde{\tau}_{y_1} \le t]\mathbf{P}[\forall s \in [0,1] : X^\pi_{\widetilde{\tau}_{y_1}+s} = y_1] \ge e^{-1}\mathbf{P}[\widetilde{\tau}_{y_1} \le t], \end{aligned} \tag{111}$$

where in the last inequality we use the strong Markov property. As the process $(X_t^\pi)_{t\ge0}$ is stationary,

$$\mathbf{E}[u(t+1)] = (t+1)\pi_{y_1,1}^\omega(y_1) \le (t+1)e^{V(y_1)-V(x_1)},$$

which allows to conclude that

$$\mathbf{P}[\widetilde{\tau}_1 \le t] \le e(t+1)e^{V(y_1)-V(x_1)}. \tag{112}$$ □

5.2. *A lower bound derived from flow consideration.* Let us now derive the third bound, which is necessary to complete the proof of Theorem 2.5.

PROPOSITION 5.2. *There exists a positive constant $c = c(\alpha, \mathbb{P})$ such that w.h.p. we have*

$$t_{\mathrm{mix}}^{N,k} \ge ckN^{\frac{1}{2\lambda}}(\log N)^{-2(1+\frac{1}{\lambda})}. \tag{113}$$

To prove the above result, we adopt the strategy developed in [35], Proposition 4.2, by investigating the flow of particles through a *slow* segment of size of order $(\log N)$ where the drift of the random environment points to the left. This flow of particles is controlled via a comparison with a boundary driven exclusion process.

In [35], the *slow* segment is selected to be such that $\omega_x < 1/2$ for *every site* in it. It has the advantage of simplifying the computation since it allows for comparison with the homogeneous exclusion process for which computation has been performed in [4]. Our approach brings an improvement by selecting the slow segment based on the potential function V^ω. The relevant quantity that limits the flow is the worst potential barrier that the particles have to overcome. Proposition 3.4 allows to identify the worst potential barrier in the system. We let $N/2 \le x_2(\omega) \le y_2(\omega) \le 3N/4$ be such that

$$V^\omega(y_2) - V^\omega(x_2) = \max_{N/2\le x\le y\le 3N/4}\big(V^\omega(y) - V^\omega(x)\big).$$

According to Proposition 3.4, we have w.h.p. (for all choices of x_2 and y_2 if there are several possibilities)

$$V(y_2) - V(x_2) \ge \frac{1}{\lambda}(\log N - 2\log\log N) \quad \text{and} \quad y_2 - x_2 \le q_N. \tag{114}$$

In order to illustrate how the mixing time can be controlled using the flow of particles, we start with a simple lemma. Let J_t denote the number of particles on the last portion of the segment,

$$J_t := \sum_{x \geq y_2+1} \sigma_t^{\min}(x). \tag{115}$$

LEMMA 5.3. *For any* $\varepsilon > 0$, *for all* $k \in [\![1, N/2]\!]$ *and for every* $t \geq 0$, *with high probability w.r.t.* $\mathbb{P}$ *we have*

$$d_{N,k}^{\omega}(t) \geq 1 - \frac{4\mathbf{E}[J_t]}{k} - \varepsilon. \tag{116}$$

PROOF. Setting $\mathcal{B} := \{\xi \in \Omega_{N,k} : \sum_{x \geq y_2+1} \xi(x) < k/4\}$, we have

$$d_{N,k}^{\omega}(t) \geq \| P_t^{\xi_{\min}} - \pi_{N,k}^{\omega} \|_{\mathrm{TV}} \geq \mathbf{P}[\sigma_t^{\min} \in \mathcal{B}] - \pi_{N,k}^{\omega}(\mathcal{B}). \tag{117}$$

By Lemma 2.3, the second term is smaller than ε with high probability w.r.t. $\mathbb{P}$. Concerning the first term, we have by Markov's inequality

$$\mathbf{P}[\sigma_t^{\min} \in \mathcal{B}] = 1 - \mathbf{P}[J_t \geq k/4] \geq 1 - \frac{4\mathbf{E}[J_t]}{k}. \tag{118}$$

□

Now we can control $\mathbf{E}[J_t]$ by comparing our system with one in which the particles flow faster. We consider the state space

$$\widetilde{\Omega}_{x_2,y_2} := \{\xi : [\![x_2, y_2+1]\!] \to \mathbb{Z}_+ : \forall x \in [\![x_2, y_2]\!], \xi(x) \in \{0,1\}\}, \tag{119}$$

and define an alternative process on $\widetilde{\Omega}_{x_2,y_2}$. There is no conservation of the number of particles in this process: the particles follow the exclusion dynamics in the bulk but new rules are added at the boundaries. If $\xi(x_2) = 0$, then a particle is added at site x_2 with rate w_{x_2-1}. At the other end of the segment, particles can jump from site y_2 to site $y_2 + 1$ without respecting the exclusion rule (i.e., the site $y_2 + 1$ is allowed to contain arbitrarily many particles) and particles at site $y_2 + 1$ remain there forever. We define the generator of the process to be (for $f : \widetilde{\Omega}_{x_2,y_2} \mapsto \mathbb{R}$)

$$\begin{aligned}(\widetilde{\mathfrak{L}}_{x_2,y_2}^{\omega} f)(\xi) := {} & \sum_{z=x_2}^{y_2-1} r^{\omega}(\xi, \xi^{z,z+1})[f(\xi^{z,z+1}) - f(\xi)] \\ & + \omega_{x_2-1}\mathbf{1}_{\{\xi(x_2)=0\}}[f(\xi + \delta_{x_2}) - f(\xi)] \\ & + \omega_{y_2}\mathbf{1}_{\{\xi(y_2)=1\}}[f(\xi - \delta_{y_2} + \delta_{y_2+1}) - f(\xi)],\end{aligned} \tag{120}$$

where r^{ω} is defined in (10). We refer to Figure 6 for a graphical description. We let $(\widetilde{\sigma}_t^{\xi})_{t\geq 0}$ denote the corresponding process starting from an initial condition $\xi \in \widetilde{\Omega}_{x_2,y_2}$.

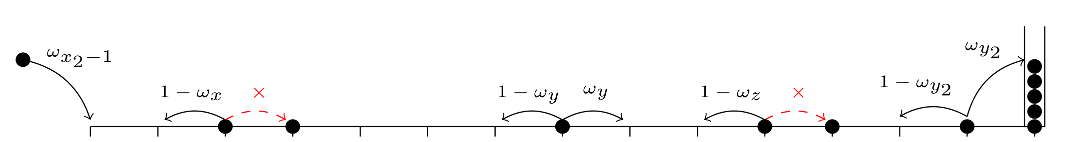

FIG. 6. *A graphical representation of the boundary driven process: a bold circle represents a particle, and the number above every arrow represents the jump rate while a red "×" represents a nonadmissible jump. In addition, the site* $y_2 + 1$ *can accommodate infinite many particles and all particles at site* $y_2 + 1$ *stay put.*

LEMMA 5.4. *Let* $\mathbf{0}$ *denote the configuration with all sites in* $[\![x_2, y_2+1]\!]$ *being empty, and let* $(\widetilde{\sigma}_t^{\mathbf{0}})_{t\geq 0}$ *denote the chain associated with the generator* $\widetilde{\mathfrak{L}}^{\omega}_{x_2,y_2}$ *starting from* $\mathbf{0}$. *Then we have*

$$J_t \leq \widetilde{\sigma}_t^{\mathbf{0}}(y_2+1), \tag{121}$$

where J_t *is defined in* (115).

PROOF. Note that for a fixed ω, the two processes $(\widetilde{\sigma}_t^{\mathbf{0}})_{t\geq 0}$ and $(\sigma_t^{\min})_{t\geq 0}$ share the same jump rates in the interval $[\![x_2, y_2]\!]$ as can be seen from the comparison of Figure 6 with Figure 1. The process $(\widetilde{\sigma}_t^{\mathbf{0}})_{t\geq 0}$ can be constructed together with $(\sigma_t^{\min})_{t\geq 0}$ on the same probability space using the graphical construction of Section 3.2 with the same clocks $(T_n^{(x)})_{x,n\in\mathbb{N}}$ and auxiliary variables $(U_n^{(x)})_{x,n\in\mathbb{N}}$ for both processes (with the obvious adaptation of the construction to fit the boundary conditions for $(\widetilde{\sigma}_t^{\mathbf{0}})_{t\geq 0}$). It can then be checked by inspection that under this coupling, for every $t \geq 0$,

$$\forall x \in [\![x_2, y_2+1]\!] \quad \sum_{z=x}^{N} \sigma_t^{\min}(z) \leq \sum_{z=x}^{y_2+1} \widetilde{\sigma}_t^{\mathbf{0}}(z). \tag{122}$$

Since the above inequality is satisfied at $t = 0$, it is sufficient to check that it is conserved by any update of the two processes. The result then just corresponds to the case $x = y_2 + 1$. $\square$

The next step is to evaluate the rate at which particles flow through the trap for the boundary driven process. The following result shows that it can be upper bounded by $e^{-\Delta V_{\max}/2(1+o(1))}$, where $\Delta V_{\max}$ is the largest potential barrier. We refer to Figure 7 for a heuristic explanation.

PROPOSITION 5.5. *There exists a constant* $C = C(\alpha, \mathbb{P})$ *such that for all* $t \geq 0$ *w.h.p. we have*

$$\mathbf{E}\big[\widetilde{\sigma}_t^{\mathbf{0}}(y_2+1)\big] \leq tCN^{-\frac{1}{2\lambda}}(\log N)^{2(1+\frac{1}{\lambda})}. \tag{123}$$

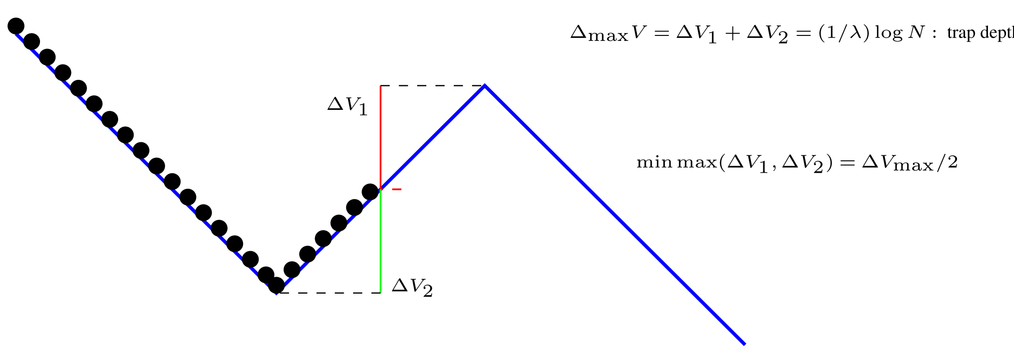

FIG. 7. *We represent a particle configuration around the potential barrier. The height of a site corresponds to its potential* (*and we have drawn* V *as a piecewise affine function for simplicity*). *Due to the potential slope, particles tend to accumulate on the left-hand side of the trap. Since particles partially fill the trap, the effective potential barrier for particles to overcome to exit the trap is smaller than* $\Delta V_{\max}$ (*which is the barrier that a single particle would have to overcome*). *It is equal to* ΔV_1, *the potential difference between the right-most particle and the right end of the trap. The typical time needed for a particle to escape the trap is thus* $\exp(\Delta V_1)$. *On the other hand, when a particle exits the trap, it must be replaced by a particle coming from the left to maintain the flow. In other words, an empty site must exit the trap on the left, and by symmetry this takes a typical time* $\exp(\Delta V_2)$, *where* $\Delta V_2 = \Delta V_{\max} - \Delta V_1$ *is the corresponding energy barrier. The flow of particles is maximized when the rates at which particles enter and exit the trap are equal, and thus, in its "steady state" the trap is "half-filled" with particles and* $\Delta V_1 = \Delta V_2 = \Delta V_{\max}/2$. *The time for a particle to travel through the trap is thus given by* $\exp(\Delta V_{\max}/2)$. *Our proof transforms this heuristic into a rigorous upper bound for the flow of particles.*

With Proposition 5.5, whose proof is detailed in the next subsection, we are ready to conclude the proof of Proposition 5.2.

PROOF OF PROPOSITION 5.2. By Lemma 5.3 and Lemma 5.4, w.h.p. for $\mathbb{P}$ we have

$$d^{\omega}_{N,k}(t) \geq \frac{7}{8} - 4\frac{\mathbf{E}[\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1)]}{k}. \tag{124}$$

In view of Proposition 5.5, we can take

$$t = \frac{1}{8C} k N^{\frac{1}{2\lambda}} (\log N)^{-2(1+\frac{1}{\lambda})}$$

in (124) to conclude the proof. □

5.3. *Proof of Proposition* 5.5. Note that $\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1)$ is a superadditive ergodic sequence. To see this, we let ϑ_s denote the time shift operator on the graphical construction variables. Recalling (46), we set

$$\vartheta_s\big((T^{(x)}_i, U^{(x)}_i)_{x\in\mathbb{Z}, i\geq 1}\big) := \big(T^{(x)}_{i+i_0(x,s)} - s, U^{(x)}_{i+i_0(x,s)}\big)_{x\in\mathbb{Z}, i\geq 1}. \tag{125}$$

Now we observe that the graphical construction preserves the order $\preccurlyeq$ on $\widetilde{\Omega}_{x_2,y_2}$ defined by

$$\xi \preccurlyeq \xi' \quad \text{if } \forall x \geq x_2, \qquad \sum_{z=x}^{y_2+1} \xi(z) \leq \sum_{z=x}^{y_2+1} \xi'(z). \tag{126}$$

Hence, comparing the dynamic in the interval $[s, s+t]$ with that starting from $\mathbf{0}$ at time s, we obtain that

$$\widetilde{\sigma}^{\mathbf{0}}_{s+t}(y_2+1) \geq \widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) + (\vartheta_s \circ \widetilde{\sigma})^{\mathbf{0}}_t(y_2+1). \tag{127}$$

Since the time-shift operator ϑ_s on (T, U) is ergodic, using (127) and $\mathbb{E}[\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1)] \leq t$ we can apply Kingman's subbadditive ergodic Theorem [20] (continuous time version) to obtain

$$\mathbf{E}\big[\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1)\big] \leq t\bigg[\lim_{s\to\infty} \frac{1}{s}\widetilde{\sigma}^{\mathbf{0}}_s(y_2+1)\bigg]. \tag{128}$$

Letting $\mathcal{N}_s := \sum_{x=x_2}^{y_2} \widetilde{\sigma}^{\mathbf{0}}_s(x)$ denote the number of mobile particles in the system (particles at site y_2+1 which have stopped moving are not counted), we have

$$\widetilde{\sigma}^{\mathbf{0}}_t(y_2+1) = \sum_{s\in(0,t]} \mathbf{1}_{\{\mathcal{N}_s < \mathcal{N}_{s\_}\}}. \tag{129}$$

Letting $(\mathcal{T}_n)_{n\geq 1}$ denote the sequence of times at which $\mathcal{N}_t < \mathcal{N}_{t\_}$ (in increasing order), we have

$$\lim_{s\to\infty} \frac{1}{s}\widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) = \lim_{n\to\infty} \frac{n}{\mathcal{T}_n}. \tag{130}$$

By evacuating all the sites in $[\![x_2, y_2]\!]$ and adding $y_2 - x_2 + 1$ particles at site y_2+1, we obtain that for any $s \geq 0$ (recall (126))

$$\widetilde{\sigma}^{\mathbf{0}}_s \preccurlyeq \big[\widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) + y_2 - x_2 + 1\big]\mathbf{1}_{\{y_2+1\}}. \tag{131}$$

Note that every site in $[\![x_2, y_2]\!]$ in the configuration of the right-hand side of (131) is empty. Running the process for an additional time period of length t on both sides of (131), we obtain (as a consequence of order preservation)

$$\widetilde{\sigma}^{\mathbf{0}}_{s+t}(y_2+1) \leq \widetilde{\sigma}^{\mathbf{0}}_s(y_2+1) + (\vartheta_s \circ \widetilde{\sigma})^{\mathbf{0}}_t(y_2+1) + y_2 - x_2 + 1. \tag{132}$$

Now as a consequence of (132) we obtain that for any $l > y_2 - x_2 + 2$,

$$\mathcal{T}_l \geq \mathcal{T}_{l-(y_2-x_2+2)} + \vartheta_{\mathcal{T}_l} \circ \mathcal{T}_1. \tag{133}$$

Since $\mathcal{T}_l$ is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$ (recall (47)), by the strong Markov property $\vartheta_{\mathcal{T}_l} \circ \mathcal{T}_1$ is independent of $\mathcal{T}_1$ and has the same distribution. Iterating the process starting with $l = (r-1)(y_2 - x_2 + 2) + 1$, we obtain

$$\mathcal{T}_{(r-1)(y_2-x_2+2)+1} \geq \mathcal{T}_1^{(1)} + \cdots + \mathcal{T}_1^{(r)}, \tag{134}$$

where $(\mathcal{T}_1^{(a)})_{a=1}^r$ is a sequence of IID copies of $\mathcal{T}_1$. This yields that

$$\liminf_{n\to\infty} \frac{\mathcal{T}_n}{n} \geq \frac{1}{y_2 - x_2 + 2} \mathbf{E}[\mathcal{T}_1]. \tag{135}$$

Finally, let us compare $(\widetilde{\sigma}_t^{\mathbf{0}})_{t\geq 0}$ with $(\widetilde{\sigma}_t')_{t\geq 0}$ starting from another initial condition, which we now specify. Let us first choose the number of particles by setting

$$\begin{aligned} &\Lambda(\omega) := \{x \in [\![x_2, y_2]\!] : V(x) \leq [V(y_2) + V(x_2)]/2\}, \\ &k'(\omega) := \#\Lambda(\omega). \end{aligned} \tag{136}$$

We let $(\widetilde{\sigma}_t')_{t\geq 0}$ be the dynamic with generator (120) and its initial configuration $\widetilde{\sigma}_0'$ is obtained by setting $\widetilde{\sigma}_0'(y_2 + 1) = 0$ and sampling $(\widetilde{\sigma}_0'(x))_{x\in[\![x_2,y_2]\!]}$ from the invariant probability measure for the exclusion process on the segment $[\![x_2, y_2]\!]$ with k' particles and environment $(\omega_x)_{x\in[\![x_2,y_2]\!]}$ (we denote this probability measure by $\pi^{\omega}_{[x_2,y_2],k'}$). Note that $\pi^{\omega}_{[x_2,y_2],k'}$ is not the projection of the invariant probability measure for the chain with generator $\widetilde{\mathfrak{L}}^{\omega}_{x_2,y_2}$, defined in (120), projected onto the segment $[\![x_2, y_2]\!]$. Using monotonicity again, we have

$$\mathcal{T}_1 \geq \inf\{t \geq 0 : \widetilde{\sigma}_t'(y_2 + 1) = 1\} \geq \inf\{t \geq 0 : \widetilde{\sigma}_t'(x_2) = 0 \text{ or } \widetilde{\sigma}_t'(y_2) = 1\} =: \mathcal{T}'. \tag{137}$$

Now let us observe that until time $\mathcal{T}'$, the process $(\widetilde{\sigma}_t')_{t\geq 0}$ (or rather, its restriction to $[\![x_2, y_2]\!]$) coincides with the exclusion process on the segment $[\![x_2, y_2]\!]$ with k' particles. Using this, we can prove the following (the proof is postponed to the end of the section).

LEMMA 5.6. *We have*

$$\mathbf{E}[\mathcal{T}'] \geq \frac{1}{16e^2(y_2 - x_2)} e^{\frac{V(y_2)-V(x_2)}{2}}. \tag{138}$$

Let us now conclude the proof of Proposition 5.5. Combing (128), (130), (135) and (137), for all $t \geq 0$ we have

$$\mathbf{E}[\widetilde{\sigma}_t^{\mathbf{0}}(y_2 + 1)] \leq t\Big[\lim_{s\to\infty} \frac{1}{s}\widetilde{\sigma}_s^{\mathbf{0}}(y_2 + 1)\Big] \leq \frac{t(y_2 - x_2 + 2)}{\mathbf{E}[\mathcal{T}_1]} \leq \frac{t(y_2 - x_2 + 2)}{\mathbf{E}[\mathcal{T}']}. \tag{139}$$

Using Lemma 5.6, we obtain

$$\mathbf{E}[\widetilde{\sigma}_t^{\mathbf{0}}(y_2 + 1)] \leq t16e^2(y_2 - x_2 + 2)^2 e^{-\frac{V(y_2)-V(x_2)}{2}}. \tag{140}$$

By (114), we have w.h.p.,

$$\mathbf{E}[\widetilde{\sigma}_t^{\mathbf{0}}(y_2 + 1)] \leq t16e^2(q_N + 2)^2 N^{-\frac{1}{2\lambda}} (\log N)^{\frac{1}{\lambda}}. \tag{141}$$

PROOF OF LEMMA 5.6. With a small abuse of notation, in this proof $(\widetilde{\sigma}_t')_{t\geq 0}$ denotes the exclusion process on the segment $[\![x_2, y_2]\!]$ with k' particles starting from the stationary distribution $\pi^{\omega}_{[x_2,y_2],k'}$. Since $\mathbf{E}[\mathcal{T}'] \geq t\mathbf{P}[\mathcal{T}' > t]$, our goal is to provide a lower bound on $\mathbf{P}[\mathcal{T}' > t]$ for some well-chosen $t > 0$. We define

$$\begin{aligned} \mathcal{B}_1 &:= \{\xi \in \Omega_{[\![x_2,y_2]\!],k'} : \xi(x_2) = 0\}, \\ \mathcal{B}_2 &:= \{\xi \in \Omega_{[\![x_2,y_2]\!],k'} : \xi(y_2) = 1\}. \end{aligned} \tag{142}$$

Using the strong Markov property at $\mathcal{T}'$ and the fact that jumping rates for particles are bounded from above by one at every site, we have

$$\mathbf{P}[\forall t \in [\mathcal{T}', \mathcal{T}' + 1], \tilde{\sigma}'_t \in \mathcal{B}_1 \cup \mathcal{B}_2] \geq e^{-2}.$$

Using independence as in (111), we have

$$\mathbf{P}[\mathcal{T}' \leq t] \leq e^2(t+1)\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_1 \cup \mathcal{B}_2). \tag{143}$$

We now head to provide an upper bound on $\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_1)$. Recalling the definition of Λ in (136), we observe that when $\xi \in \mathcal{B}_1$, since $x_2 \in \Lambda$ and there are k' particles, there must be a particle in $\Lambda^{\complement} := [\![x_2, y_2]\!] \setminus \Lambda$. Let $\mathtt{R}(\xi)$ be the position of the rightmost such particle

$$\mathtt{R}(\xi) := \sup\{z \in \Lambda^{\complement} : \xi(z) = 1\},$$

and set for $z \in \Lambda^{\complement}$

$$\mathcal{B}_{1,z} := \{\xi \in \mathcal{B}_1 : \mathtt{R}(\xi) = z\}.$$

By moving the particle from site z to site x_2 as in (72), we obtain

$$\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_{1,z}) = \sum_{\xi \in \mathcal{B}_{1,z}} \pi^{\omega}_{[x_2,y_2],k'}(\xi^{x_2,z})e^{-V(z)+V(x_2)} \leq e^{-V(z)+V(x_2)} \leq e^{-\frac{V(y_2)-V(x_2)}{2}},$$

and then

$$\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_1) = \sum_{z \in \Lambda^{\complement}} \pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_{1,z}) \leq (y_2 - x_2)e^{-\frac{V(y_2)-V(x_2)}{2}}. \tag{144}$$

Similarly, we can obtain

$$\pi^{\omega}_{[x_2,y_2],k'}(\mathcal{B}_2) \leq (y_2 - x_2)e^{-\frac{V(y_2)-V(x_2)}{2}}. \tag{145}$$

Combining (144) with (145), in (143) we take

$$t = \frac{1}{4e^2(y_2 - x_2)} e^{\frac{V(y_2)-V(x_2)}{2}} - 1$$

to obtain

$$\mathbf{E}[\mathcal{T}'] \geq \frac{1}{2}\left(\frac{1}{4e^2(y_2 - x_2)} e^{\frac{V(y_2)-V(x_2)}{2}} - 1\right) \geq \frac{1}{16e^2(y_2 - x_2)} e^{\frac{V(y_2)-V(x_2)}{2}}. \tag{146}$$ □

6. Upper bound on the mixing time. This section is dedicated to the proof of Theorem 2.4. First, in Section 6.1 we are going to reduce the problem to an estimate of the transition probability between extremal states, which is $P_t(\xi_{\min}, \xi_{\max})$. Next, by the censoring scheme and particle transport (cf. Propositions 3.2 and 3.3) we are going to estimate this probability. For pedagogical reason, we first treat the simpler case where the number of particles is small and for which only censoring (that is Proposition 3.2) is needed. This is the case $k \leq q_N$ (recall (59), which is treated in Section 6.2. The general case $q_N < k \leq N/2$, which relies on Proposition 3.3, is then treated in Section 6.4.

Let us introduce notation for the exclusion process with k particles on the segment $[\![a, b]\!]$ with environment ω for arbitrary integers $a \leq b$, which are used in this section. We let $\Omega_{[a,b],k}$ denote the corresponding state space, $\pi^{\omega}_{[a,b],k}$ denote the the equilibrium measure and $d^{\omega}_{[a,b],k}(t)$ denote the total variation distance to equilibrium (from worst starting position, cf. (15)).

6.1. *Deducing the mixing time from the hitting time of the maximal configuration.* Let us first show that the study of the mixing time can be reduced to that of the probability of hitting the configuration $\xi_{\max}$ starting from the other extremal configuration $\xi_{\min}$.

PROPOSITION 6.1. *We have for every $t > 0$ and $n \in \mathbb{N}$,*

$$d^{\omega}_{N,k}(nt) \le \big(1 - P_t(\xi_{\min}, \xi_{\max})\big)^n. \tag{147}$$

PROOF. We have (see, for instance, [27], Lemma 4.10)

$$d^{\omega}_{N,k}(t) \le \bar{d}^{\omega}_{N,k}(t) := \max_{\xi,\xi'} \| P^{\xi}_t - P^{\xi'}_t \|_{\mathrm{TV}} \le \max_{\xi,\xi'} \mathbf{P}\big[\sigma^{\xi}_t \neq \sigma^{\xi'}_t\big]. \tag{148}$$

Using the monotonicity under the graphical construction (cf. Proposition 3.1) for all $\xi \in \Omega_{N,k}$ and $t \ge 0$, we have

$$\sigma^{\min}_t \le \sigma^{\xi}_t \le \sigma^{\max}_t,$$

where $(\sigma^{\min}_t)_{t\ge 0}$ and $(\sigma^{\max}_t)_{t\ge 0}$ are starting from the extremal conditions $\xi_{\min}$ and $\xi_{\max}$ in (45). As a consequence for arbitrary ξ and ξ', setting $\tau' := \inf\{t \ge 0 : \sigma^{\min}_t = \sigma^{\max}_t\}$, we have

$$\forall t \ge \tau', \quad \sigma^{\xi}_t = \sigma^{\xi'}_t. \tag{149}$$

On the other hand, we have

$$\tau' \le \tau := \inf\{t \ge 0 : \sigma^{\min}_t = \xi_{\max}\}. \tag{150}$$

Therefore, (148) implies that

$$d^{\omega}_{N,k}(t) \le \mathbf{P}(\tau > t). \tag{151}$$

Using again the Markov property and the monotonicity in Proposition 3.1, we have for any positive integer n,

$$\mathbf{P}(\tau > nt) \le \mathbf{P}\big(\sigma^{\min}_{it} \neq \xi_{\max}, \forall i \in [\![1, n]\!]\big) \le \mathbf{P}\big(\sigma^{\min}_t \neq \xi_{\max}\big)^n. \tag{152}$$ □

6.2. *The case $k_N \le q_N$.* Before stating the main result of this section, let us present a strategy to bound $P_t(\xi_{\min}, \xi_{\max})$ from below. We present in the process a few key technical lemmas whose proofs are presented in the next subsection. We consider environments in the following set:

$$\mathcal{A}_N := \Big\{\omega : \max_{\substack{1\le x\le y\le N\\ y-x\ge q_N}} \big(V(y) - V(x)\big) \le -3\log N\Big\}. \tag{153}$$

Note that by Proposition 3.4, $\mathcal{A}_N$ is an high probability event. This condition ensures that when considering the exclusion process restricted to subsegments of $[\![1, N]\!]$ of length $4q_N$, at equilibrium the particles typically concentrate on the right half of the segment. If the number of particles is large enough, it also ensures that typically at equilibrium the last site is occupied by a particle. This is the content of our first technical lemma.

LEMMA 6.2. *If $\omega \in \mathcal{A}_N$, then we have for any $x \in [\![0, N - 4q_N]\!]$ and any $k \le q_N$,*

$$\begin{aligned} &\pi^{\omega}_{[x+1,x+4q_N],k}\big[\bar{\xi}(1) \le x + 2q_N\big] \le 2q_N^2 N^{-3},\\ &\pi^{\omega}_{[x+1,x+4q_N],q_N}\big[\xi(x + 4q_N) = 0\big] \le 3q_N N^{-3}. \end{aligned} \tag{154}$$

Our second technical lemma is a direct consequence of Proposition 2.2. It allows to bound the mixing time of the system for each of the intervals of length $4q_N$ in a quantitative way. We define

$$T = T_N := 80\alpha^{-1} q_N^4 \binom{4q_N}{q_N} \left(\frac{1-\alpha}{\alpha}\right)^{2q_N} \log\left(\frac{1-\alpha}{\alpha}\right). \tag{155}$$

The following result is obtained by taking $\varepsilon = N^{-3}$ in Proposition 2.2.

LEMMA 6.3. *Under the assumption* (3), *for all N sufficiently large we have for for all $k \le q_N$, all $x \in [\![0, N-4q_N]\!]$ and almost every realization of ω,*

$$d^{\omega}_{[x+1,x+4q_N],k}(T) \le N^{-3}. \tag{156}$$

Our third technical lemma ensures that in typical environment, the weight of $\xi_{\max}$ at equilibrium is not too small. In the definition of $\mathcal{B}_{N,k}$ below, $\xi_{\max}$ denotes (with a small abuse of notation) the maximal configuration with k particles in the segment $[\![N-4q_N+1, N]\!]$.

LEMMA 6.4. *We have*

$$\lim_{\varepsilon\to 0} \inf_{\substack{N\ge 1\\ k\in[\![1,N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}(\xi_{\max}) > \varepsilon\right] = 1. \tag{157}$$

In particular, if $\mathcal{B}_{N,k} := \{\omega : \pi^{\omega}_{[N-4q_N+1,N],k}(\xi_{\max}) \ge 2q_N^{-1}\}$, we have

$$\lim_{N\to\infty} \inf_{k\in[\![1,q_N]\!]} \mathbb{P}[\mathcal{B}_{N,k}] = 1. \tag{158}$$

Now that the technical prelimiaries are set, we can introduce the main technical result proved in this section.

PROPOSITION 6.5. *If $k \le q_N$, if $\omega \in \mathcal{A}_N \cap \mathcal{B}_{N,k}$ and $t_0 := T(\lceil \frac{N}{2q_N}\rceil - 1)$, we have*

$$P_{t_0}(\xi_{\min}, \xi_{\max}) \ge \frac{3}{2q_N}. \tag{159}$$

In particular, the inequality holds with high probability w.r.t. the environment law $\mathbb{P}$.

The last part of the statement is of course a direct consequence of the first part combined with (158) and of Proposition 3.4 (which ensures that $\mathcal{A}_N$ and $\mathcal{B}_{N,k}$ are high probability events). Before providing a proof of Proposition 6.5, let us show how it implies the desired upper bound on the mixing time.

PROOF OF THEOREM 2.4 WHEN $k \le q_N$. By Proposition 6.1 and Proposition 6.5, we have

$$d^{\omega}_{N,k}(2q_N t_0) \le \left(1 - P_{t_0}(\xi_{\min}, \xi_{\max})\right)^{2q_N} \le \left(1 - \frac{1}{q_N}\right)^{2q_N} \le \frac{1}{4}, \tag{160}$$

which allows us to conclude the proof for the case $k \le q_N$ with the inequality

$$\binom{4q_N}{q_N} \le \left(\frac{4^4}{3^3}\right)^{q_N}.$$

□

Let us briefly sketch now our proof of Proposition 6.5. We use Proposition 3.2 to channel all the particles to the right. More precisely, we use a censoring scheme that during time interval $[iT,(i+1)T)$, with $i=0,\dots,\lceil N/(2q_N)\rceil-3$ isolates the segment $[\![2iq_N+1,2(i+2)q_N]\!]$ (of length $4q_N$) from the rest of the system simply by censoring the transitions along edges at the extremity of this segment; cf. (161)–(162).

In Lemma 6.6, using a combination of Lemmas 6.2 and Lemma 6.3 we show that with high probability, at all times, under the censored dynamics, all the particles remain within this "traveling isolated segment," which eventually channels all particles to the segment $[\![N-4q_N+1,N]\!]$ at time $(\lceil N/(2q_N)\rceil-2)T$. Once this is done, combining Lemmas 6.3 and 6.4, we show that with probability at least $3/(2q_N)$ we end up at $\xi_{\max}$ within an additional time T.

We introduce our censoring scheme and define (in the middle line $i\in[\![1,\lceil N/(2q_N)\rceil-3]\!]$)

$$
\begin{cases}\mathcal{C}_0:=\{\{4q_N,4q_N+1\}\},\\ \mathcal{C}_i:=\{\{i2q_N,i2q_N+1\},\{(i+2)2q_N,(i+2)2q_N+1\}\},\\ \mathcal{C}_{\lceil N/(2q_N)\rceil-2}:=\{\{N-4q_N,N-4q_N+1\}\}.\end{cases}\tag{161}
$$

We define a censoring scheme by setting (recall that T is defined in (155))

$$
\mathcal{C}(t):=\mathcal{C}_i\quad \text{for } t\in[iT,(i+1)T), i\in[\![0,\lceil N/(2q_N)\rceil-2]\!],\tag{162}
$$

and $\mathcal{C}(t)=\varnothing$ for $t\ge(\lceil N/(2q_N)\rceil-1)T$. Let us write

$$
A_{\mathrm{fin}}:=\{\xi\in\Omega_{N,k}:\forall x\in[\![1,N-4q_N]\!],\xi(x)=0\}.\tag{163}
$$

Recalling the notation of Section 3.3, we let $(\sigma_t^{\min,\mathcal{C}})_{t\ge0}$ denote the corresponding censored dynamics with initial condition $\xi_{\min}$.

LEMMA 6.6. *If $\omega\in\mathcal{A}_N$, we have*

$$
\mathbf{P}[\sigma^{\min,\mathcal{C}}_{(\lceil N/2q_N\rceil-2)T}\in A_{\mathrm{fin}}]\ge 1-N^{-1}.\tag{164}
$$

PROOF. For $i\in[\![0,\lceil N/2q_N\rceil-2]\!]$, we define A_i to be the set of configurations for which all particles lie in the interval $[\![2iq_N+1,2(i+2)q_N]\!]$,

$$
A_i:=\{\xi\in\Omega_{N,k}:2iq_N<\bar{\xi}(1)\le\bar{\xi}(k)\le 2(i+2)q_N\}.
$$

Now we prove by induction that the probability that particles remain trapped in the segment $[\![2iq_N+1,2(i+2)q_N]\!]$ during time interval $[iT,(i+1)T)$ is high, which is

$$
\mathbf{P}[\sigma^{\min,\mathcal{C}}_{iT}\in A_i]\ge 1-i\frac{4q_N^2}{N^3}.\tag{165}
$$

Since $2(\lceil N/2q_N\rceil-2)q_N\ge N-4q_N$, the result (164) follows from the case $i=\lceil N/2q_N\rceil-2$ in (165). From the definition of $\xi_{\min}$, the inequality in (165) holds for $i=0$. Assuming that (165) holds for i, then from our choice of censoring scheme, during the time interval $[iT,(i+1)T)$ the k particles perform the simple exclusion process on the segment $[\![2iq_N+1,2(i+2)q_N]\!]$. By Lemma 6.2 and Lemma 6.3 with $x=2iq_N$, we have

$$
\begin{aligned}
&\mathbf{P}[\sigma^{\min,\mathcal{C}}_{(i+1)T}\in A_{i+1}]\\
&\quad\ge \mathbf{P}[\sigma^{\min,\mathcal{C}}_{iT}\in A_i]\\
&\qquad-(\pi^{\omega}_{[2iq_N+1,2(i+2)q_N],k}(\bar{\xi}(1)\le 2(i+1)q_N)+d^{\omega}_{[2iq_N+1,2(i+2)q_N],k}(T))\\
&\quad\ge 1-i\frac{4q_N^2}{N^3}-\frac{4q_N^2}{N^3}.
\end{aligned}\tag{166}
$$

□

PROOF OF PROPOSITION 6.5. Using Proposition 3.2, it is sufficient to bound the corresponding probability for the censored dynamics, that is,

$$P^{\mathcal{C}}_{(\lceil N/2q_N\rceil-1)T}(\xi_{\min},\xi_{\max}).$$

If $\sigma^{\min,\mathcal{C}}_{(\lceil N/2q_N\rceil-2)T}\in A_{\mathrm{fin}}$, then due to our choice of censuring scheme during the time interval $[\lceil N/2q_N\rceil-2T,(\lceil N/2q_N\rceil-1)T)$, the dynamics corresponds to an exclusion process with k particles on the segment $[\![N-4q_N+1,N]\!]$. We have

$$\begin{aligned}&P^{\mathcal{C}}_{(\lceil N/2q_N\rceil-1)T}(\xi_{\min},\xi_{\max})\\ &\quad\ge \mathbf{P}\big[\sigma^{\xi_{\min},\mathcal{C}}_{(\lceil N/2q_N\rceil-2)T}\in A_{\mathrm{fin}}\big]\big(\pi^{\omega}_{[N-4q_N+1,N],k}(\xi_{\max})-d^{\omega}_{[N-4q_N+1,N],k}(T)\big)\\ &\quad\ge \big(1-N^{-1}\big)\big(2q_N^{-1}-N^{-3}\big)\ge\frac{3}{2q_N},\end{aligned}\tag{167}$$

where we have used the definition of $\mathcal{B}_{N,k}$ (recall (158)) and Lemma 6.3 with $x=N-4q_N$. □

6.3. *Proof of auxiliary lemmas.*

PROOF OF LEMMA 6.2. To provide an upper bound on

$$\pi^{\omega}_{[x+1,x+4q_N],k}\big[\bar{\xi}(1)\le x+2q_N\big],$$

for $\xi\in\Omega_{[x+1,x+4q_N],k}$ we define $\bar{R}(\xi)$ to be the rightmost empty site

$$\bar{R}(\xi):=\sup\{y\in[\![x+1,x+4q_N]\!]:\xi(y)=0\}.\tag{168}$$

As in (72), for all $y,z\in[\![x+1,x+4q_N]\!]$ satisfying $y-z\ge q_N$ and $\omega\in\mathcal{A}_N$ we have

$$\pi^{\omega}_{[x+1,x+4q_N],k}\big[\bar{\xi}(1)=z,\bar{R}(\xi)=y\big]\le e^{V^{\omega}(y)-V^{\omega}(z)}\le N^{-3}.\tag{169}$$

Then we have

$$\begin{aligned}&\pi^{\omega}_{[x+1,x+4q_N],k}\big[\bar{\xi}(1)\le x+2q_N\big]\\ &\quad=\sum_{\substack{z\in[\![x+1,x+2q_N]\!]\\ y\in[\![x+4q_N-k+2,x+4q_N]\!]}}\pi_{[x+1,x+4q_N],k}\big[\bar{\xi}(1)=z,\bar{R}(\xi)=y\big]\le 2q_N^2N^{-3}.\end{aligned}\tag{170}$$

Now we estimate $\pi^{\omega}_{[x+1,x+4q_N],q_N}[\xi(x+4q_N)=0]$. For $\xi\in\Omega_{[x+1,x+4q_N],q_N}$, we define its leftmost particle to be

$$\bar{L}(\xi):=\inf\{y\in[\![x+1,x+4q_N]\!]:\xi(y)=1\}.$$

As in (72), we have

$$\pi^{\omega}_{[x+1,x+4q_N],q_N}\big[\xi(x+4q_N)=0;\bar{L}(\xi)=y\big]\le e^{V^{\omega}(x+4q_N)-V^{\omega}(y)}\le N^{-3},\tag{171}$$

where we have used $y\le x+3q_N$ and $\omega\in\mathcal{A}_N$. Then

$$\begin{aligned}&\pi^{\omega}_{[x+1,x+4q_N],q_N}\big[\xi(x+4q_N)=0\big]\\ &\quad=\sum_{y\in[\![x+1,x+3q_N]\!]}\pi^{\omega}_{[x+1,x+4q_N],q_N}\big[\xi(x+4q_N)=0;\bar{L}(\xi)=y\big]\le 3q_NN^{-3}.\end{aligned}\tag{172}$$

□

PROOF OF LEMMA 6.4. Recall the event $\mathcal{A}_r$ in (27). Observe that

$$\max_{\xi\in\mathcal{A}_r}\big(V^{\omega}(\xi_{\max})-V^{\omega}(\xi)\big)\le 2r^2\log\frac{1-\alpha}{\alpha},\tag{173}$$

and then we have

$$\frac{\pi^{\omega}_{N,k}(\xi_{\max})}{\pi^{\omega}_{N,k}(\mathcal{A}_r)} \geq |\mathcal{A}_r|^{-1} \exp\Big(-\max_{\xi\in\mathcal{A}_r}\big(V^{\omega}(\xi_{\max}) - V^{\omega}(\xi)\big)\Big) \geq 2^{-2r} e^{-2r^2 \log \frac{1-\alpha}{\alpha}}. \tag{174}$$

For given $\varepsilon \in (0, \frac{1}{2})$ sufficiently small, we take

$$r(\varepsilon) := \left\lfloor \left(\frac{-\log(2\varepsilon)}{2(1+\log\frac{1-\alpha}{\alpha})}\right)^{1/2} \right\rfloor \tag{175}$$

so that the rightmost-hand side of (174) is larger than or equal to 2ε. Moreover, by (74) we know that

$$\lim_{r\to\infty} \inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}} \mathbb{P}\left[\pi^{\omega}_{N,k}(\mathcal{A}_r) \geq 1 - 2\frac{1-\alpha}{\alpha}\big(1-e^{\frac{\mathbb{E}[\log\rho_1]}{2}}\big)^{-2} e^{\frac{\mathbb{E}[\log\rho_1]r}{2}}\right] = 1. \tag{176}$$

Since when r is sufficiently large, we have

$$1 - 2\frac{1-\alpha}{\alpha}\big(1-e^{\frac{\mathbb{E}[\log\rho_1]}{2}}\big)^{-2} e^{\frac{\mathbb{E}[\log\rho_1]r}{2}} \geq \frac{1}{2},$$

then by (176) with r chosen as in (175) we obtain

$$\lim_{\varepsilon\to 0} \inf_{\substack{N\geq 1\\ k\in[\![1,N/2]\!]}} \mathbb{P}\big[\pi^{\omega}_{N,k}(\xi_{\max}) \geq \varepsilon\big] = 1. \tag{177}$$

□

6.4. *The case $k_N \geq q_N$.* To treat the case of a larger number of particles, the problem with the strategy of the previous subsection is that it does not allow to channel all the k particles to the right at the same time. What we do instead is that we use the process to transport one particle to the right, and then use Proposition 3.3 to be able to move all other particles to the left and iterate the process. We largely recycle the strategy used in the previous section. In the final step as in (167), we need to deal with the leftmost q_N particles performing the exclusion process restricted to the interval $[\![N-k-3q_N+1, N-k+q_N]\!]$, and then define

$$\mathcal{B}'_{N,k} = \{\omega : \pi^{\omega}_{[N-k-3q_N+1,N-k+q_N],q_N}(\xi'_{\max}) \geq 2q_N^{-1}\},$$

where $\xi'_{\max} := \mathbf{1}_{\{N-k+1\leq x\leq N-k+q_N\}}$. By Lemma 6.4, we have

$$\lim_{N\to\infty} \inf_{k\in[\![q_N+1,N/2]\!]} \mathbb{P}\big[\mathcal{B}'_{N,k}\big] = 1. \tag{178}$$

PROPOSITION 6.7. *If $k > q_N$ and $\omega \in \mathcal{A}_N \cap \mathcal{B}'_{N,k}$, setting*

$$t_1 := \left(\left\lceil \frac{N-k+q_N}{2q_N}\right\rceil - 1\right)(k-q_N+1)T,$$

where T is defined in (155), we have

$$P_{t_1}(\xi_{\min}, \xi_{\max}) \geq \frac{1}{q_N}. \tag{179}$$

PROOF OF THEOREM 2.4 WHEN $k > q_N$. By Proposition 6.1 and Proposition 6.7, we have

$$d^{\omega}_{N,k}(2q_N t_1) \leq \big(1 - P_{t_1}(\xi_{\min}, \xi_{\max})\big)^{2q_N} \leq \left(1-\frac{1}{q_N}\right)^{2q_N} \leq \frac{1}{4}, \tag{180}$$

which allows us to conclude the proof for the case $k > q_N$ with the inequality

$$\binom{4q_N}{q_N} \le \left(\frac{4^4}{3^3}\right)^{q_N}. \qquad \square$$

The remainder of the subsection is devoted to the proof of Proposition 6.7. This time we combine our censoring scheme with deterministic transport of particles to the left of their current positions, and use Proposition 3.3 instead of Proposition 3.2. Transport of particles allows us to channel, one by one, the rightmost $k - q_N$ particles to the segment $[\![N - k + q_N + 1, N]\!]$ and use censoring to block these $k - q_N$ particles afterwards. We are then left with the problem of moving the remaining q_N particles to the right, and this can be treated as in Proposition 6.5.

Let us explain our plan to move the rightmost $k - q_N$ particles one by one. We proceed by induction (each step is going to leave aside an event of small probability, and our technical estimates are such that the sum of these probabilities over all steps will remain small). Let us start with the channeling of the first particle, which requires r steps with $r := \lceil (N - k + q_N)/2q_N \rceil - 1$ (each of them lasts for a time T which remains defined by (155)). During the whole process, the leftmost $k - q_N$ particles remain blocked on the leftmost sites of the segment; this can be done by constantly censoring transitions along the edge $\{k - q_N, k - q_N + 1\}$. When this is done, we are left with an effective system with q_N particles on the segment $[\![k - q_N + 1, N]\!]$ (whose length is comprised between $N/2$ and N). On this segment, we are going to apply the same technique as that for the proof of Proposition 6.5: we use censoring to maintain all these q_N particles within a subsegment of size $4q_N$ at all times. The position of this segment is shifted by an amount $2q_N$ to the right after time T. After $r - 1$ steps, we are sure that all q_N particles are on the segment $[\![N - 4q_N + 1, N]\!]$. Isolating this segment with censoring and using Lemma 6.2 (the second estimate), we can guarantee that after an additional time step, there is a particle on site N.

We can then iterate this strategy. Once we have brought j particles in the segment $[\![N - j + 1, N]\!]$ (with $1 \le j \le k - q_N$), we transport all other particles $k - j$ particles to the leftmost sites (as authorized by Proposition 3.3), and censor transitions along the edges $\{N - j, N - j + 1\}$ and $\{k - j - q_N, k - j - q_N + 1\}$. This blocks the accumulated j particles on the right, and leaves us with an effective system with q_N particles on the segment $[\![k - j - q_N + 1, N - j]\!]$, and we repeat the previous strategy to place one particle on site $N - j$ using r censoring steps.

Once $k - q_N$ particles have been brought to the right in this manner, we censor the edge $\{N - k + q_N, N - k + q_N + 1\}$ to lock these particles on the right, and conclude by repeating once more the procedure of Proposition 6.5 on the segment $[\![1, N - k + q_N]\!]$, which now contains only q_N particles.

Now that the strategy has been explained, let us write down explicitly the corresponding censoring scheme, and the transition matrices Q_j, which transport particles to the left at fixed times. Recall that $r = \lceil (N - k + q_N)/2q_N \rceil - 1$, and define for $j \in [\![0, k - q_N]\!]$, $i \in [\![0, \lceil (N - k + q_N)/2q_N \rceil - 3]\!]$, $a_{i,j} := k - q_N - j + 2q_N i$,

$$\begin{aligned} \mathcal{C}_{i,j} &:= \big\{\{k - j - q_N, k - j - q_N + 1\}, \{a_{i,j}, a_{i,j} + 1\}, \\ &\qquad \{a_{i,j} + 4q_N, a_{i,j} + 4q_N + 1\}, \{N - j, N - j + 1\}\big\}, \\ \mathcal{C}^*_j &:= \big\{\{k - j - q_N, k - j - q_N + 1\}, \{N - 4q_N - j, N - 4q_N - j + 1\}, \\ &\qquad \{N - j, N - j + 1\}\big\}. \end{aligned} \tag{181}$$

We define the censoring scheme $\mathcal{C}$ by setting

$$\begin{cases} \mathcal{C}(t)=\mathcal{C}_{i,j} & \text{if } t\in[(i+rj)T,(i+rj+1)T), \\ \mathcal{C}(t)=\mathcal{C}^*_j & \text{if } t\in[(r(j+1)-1)T, r(j+1)T), \\ \mathcal{C}(t)=\varnothing & \text{if } t\geq r(k-q_N+1)T. \end{cases} \tag{182}$$

When $j=0$ and $j=k-q_N$, the edges $\{0,1\}$ and $\{N,N+1\}$ appear in the set of censored edges described above. When these edges appear in $\mathcal{C}(t)$, this has no effect on the censoring. Then we set $s_j:=jrT$, and introduce the matrix transition Q_j, which has the effect of transporting the leftmost $k-j$ particles to the segment $[\![1,k-j]\!]$. We define thus Q_j by setting

$$Q_j(\xi,\xi^*_j)=1, \qquad Q_j(\xi,\xi')=0 \quad \text{if } \xi'\neq\xi^*_j, \tag{183}$$

where the function $\xi\to\xi^*_j$ is defined by (recall (43))

$$\bar{\xi}^*_j(\ell)=\begin{cases} \ell & \text{if } l\leq k-j, \\ \bar{\xi}(\ell) & \text{if } \ell>k-j. \end{cases} \tag{184}$$

Since $\xi^*_j\leq\xi$, Q_j satisfies (53). We let $(\widetilde{\sigma}_t)_{t\geq 0}$ denote the composed censored dynamics (recall (54)) corresponding to $\mathcal{C}$, $(s_j)_{j=1}^{k-q_N}$ and $(Q_j)_{j=1}^{k-q_N}$ and starting from $\xi_{\min}$. We set

$$\xi^0_j:=\mathbf{1}_{[\![1,k-j]\!]}+\mathbf{1}_{[\![N-j+1,N]\!]}.$$

To formalize the argument exposed above, we are going to prove by induction that at time s_j, j particles have been moved to the rightmost sites.

LEMMA 6.8. *For all $j\in[\![0,k-q_N]\!]$, we have*

$$\mathbf{P}[\widetilde{\sigma}_{rjT}=\xi^0_j]\geq 1-4jq_NN^{-2}. \tag{185}$$

REMARK 6.9. Strictly speaking, the transport of particles to the left using Q_j and the use of Proposition 3.3 instead of Proposition 3.2 are not necessary, but we felt that it resulted in a cleaner presentation.

PROOF. The statement is trivial for $j=0$. For the induction step, it is sufficient to prove that

$$\mathbf{P}[\widetilde{\sigma}_{r(j+1)T}=\xi^0_{j+1}|\widetilde{\sigma}_{rjT}=\xi^0_j]\geq 1-4q_NN^{-2}. \tag{186}$$

With our choice for $\mathcal{C}$, the j particles in the interval $[\![N-j+1,N]\!]$ do not move between time instants rjT and $r(j+1)T$, it is therefore sufficient to show that

$$\mathbf{P}[\widetilde{\sigma}_{r(j+1)T}(N-j)=1|\widetilde{\sigma}_{rjT}=\xi^0_j]\geq 1-4q_NN^{-2}. \tag{187}$$

Let us define

$$B_j:=\left\{\xi\in\Omega_{N,k}:\sum_{x=N-j-4q_N+1}^{N-j}\xi(x)=q_N\right\}. \tag{188}$$

We can repeat the proof of Lemma 6.6 to obtain that

$$\mathbf{P}[\widetilde{\sigma}_{rjT+(r-1)T}\in B_j|\widetilde{\sigma}_{rjT}=\xi^0_j]\geq 1-(r-1)\frac{4q_N^2}{N^3}. \tag{189}$$

Now in the time interval $[rjT+(r-1)T, r(j+1)T)$, the censoring makes the restriction of the dynamics to the segment $[\![N-j-4q_N+1,N-j]\!]$ an exclusion process with q_N

particles. Hence, using Lemma 6.3 and the second estimate in Lemma 6.2 we have for any $\chi \in B_j$

$$\mathbf{P}\big[\widetilde{\sigma}_{r(j+1)T}(N-j) = 1 | \widetilde{\sigma}_{rjT+(r-1)T} = \chi\big] \geq 1 - N^{-3}(1+3q_N). \tag{190}$$

Combining (189) and (190), we obtain (187). □

PROOF OF PROPOSITION 6.7. Using Proposition 3.3, it is sufficient to prove that

$$\mathbf{P}[\widetilde{\sigma}_{r(k-q_N+1)T} = \xi_{\max}] \geq \frac{1}{q_N}. \tag{191}$$

Taking $j = k - q_N$ in Lemma 6.8 and assuming that $\omega \in \mathcal{B}'_{N,k}$, we have

$$\mathbf{P}\big[\widetilde{\sigma}_{(k-q_N)rT} = \xi^0_{k-q_N}\big] \geq 1 - (k-q_N)\frac{4q_N}{N^2} \geq \frac{2}{3}. \tag{192}$$

Hence, the result follows if one can prove that

$$\mathbf{P}\big[\widetilde{\sigma}_{(k-q_N+1)rT} = \xi_{\max} | \widetilde{\sigma}_{(k-q_N)rT} = \xi^0_{k-q_N}\big] \geq \frac{3q_N}{2}. \tag{193}$$

With the conditioning, for $t \in [(k-q_N)rT, (k-q_N+1)rT)$, the rightmost $k-q_N$ particles are locked in the rightmost $k-q_N$ sites and at $t = (k-q_N)rT$ the leftmost q_N particles are in the leftmost q_N sites. We are in a similar setting as that in Proposition 6.5 with a system of q_N particles in the interval $[\![1, N-(k-q_N)]\!]$ for $t \geq (k-q_N)rT$. Thus, we can repeat the proof in Proposition 6.5 to obtain (193) (the condition $\omega \in \mathcal{B}'_{N,k}$ plays here exactly the same role as $\omega \in \mathcal{B}_{N,k}$ in the proof of Proposition 6.5). □

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